

Singular Phases of Seiberg-Witten Integrable Systems: Weak and Strong Coupling

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We consider the singular phases of the smooth finite-gap integrable systems arising in the context of Seiberg-Witten theory. These degenerate limits correspond to the weak and strong coupling regimes of SUSY gauge theories. The spectral curves in such limits acquire simpler forms: in most cases they become rational, and the corresponding expressions for coupling constants and superpotentials can be computed explicitly. We verify that in accordance with the computations from quantum field theory, the weak-coupling limit gives rise to precisely the “trigonometric” family of Calogero-Moser and open Toda models, while the strong-coupling limit corresponds to the solitonic degenerations of the finite-gap solutions. The formulae arising provide some new insights into the corresponding phenomena in SUSY gauge theories. Some open conjectures have been proven.

1 Introduction

Recent progress in understanding non-perturbative structures in supersymmetric (SUSY) gauge theories [1, 2] has also shed new insight upon the role of integrable structures in modern theoretical physics. Surprisingly, several integrable systems that were introduced and studied over the years as simplified models for quantum-mechanical and low-dimensional field theoretical problems have now reappeared as an important tool for understanding physically interesting effects in almost realistic, effective gauge theories. Part of the reason for this (familiar from supergravity investigations) is that the scalars of say a $D = 4$ $\mathcal{N} = 2$ SUSY vector supermultiplet are constrained to lie on a special Kähler manifold. With appropriate integrality constraints the cotangent bundle of such a manifold appears as the phase space of an algebraically completely integrable system (see, for example [3]). Yet an *a priori* argument for which integrable systems should arise, and what lies at the heart of this relationship between Seiberg-Witten (SW) theories [1, 2] and integrable systems [4] (see also [5, 6] and the books [7, 8] and references therein) still remains unanswered, and presents more questions than answers. Nonetheless this correspondence is accepted as providing a strong technical tool for the classification and computation of instanton effects in $\mathcal{N} = 2$ SUSY gauge theories [9, 10], with integrable systems yielding insights not yet understood in conventional gauge theoretic terms. Indeed one can straightforwardly present a *dictionary* for the correspondence between integrable systems and SW theories, but a majority of the “arrows” of the correspondence require further comment, not all of which is satisfactory.

In what follows we shall study various singular limits of this correspondence. This will yield both a better physical insight and simpler expressions for the corresponding phenomena in the quantum gauge theories. The integrable systems arising in the context of SW theory are the well-known finite-gap solutions (see, for example [11]) associated to complex curves or Riemann surfaces of finite genus. Though comparatively simple complex manifolds, explicit formulae and constructions for these integrable systems and their solutions are often lacking. However, in certain *singular* limits, their structure simplifies drastically: this happens when one comes to the boundaries of the moduli space of their complex structures, or, in different terms, to the “boundary” values of vacua parameters of the gauge theory. These are the limits we shall study in this paper. The simplifications arising at these limits will enable us to derive explicit formulae and constructions, as well as provide certain insight into the corresponding phenomena in the quantum gauge theory.

The boundaries of the moduli spaces of SW curves are characterised by either the vanishing or divergence of their period matrices T_{ij} , which play the role of couplings in the effective gauge theories. The limits $T_{ij} \rightarrow \infty$

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and $T_{ij} \rightarrow 0$ correspond to weak and strong coupling phases of the gauge theory, and they are related by the modular transformation $T \leftrightarrow T^{-1}$ playing the role of S-duality [1]. Below we shall study the corresponding limits. Both limits appear rather naturally from the perspective of the integrable models. The weak-coupling limit of the gauge theory, when instanton contributions are exponentially suppressed and can be neglected, is almost trivial from the field theory perspective. The corresponding limit on the integrable systems side is a “simplification” of the interparticle potential: the non-periodic (or open) Toda chain is a limit of the periodic Toda “molecule”; the trigonometric Calogero-Moser-Sutherland model is a degeneration of the elliptic Calogero-Moser model, and so on. In these cases (and now in fact *only* in these cases) can the SW integral formulae be practically *derived* from perturbative computations in the $\mathcal{N} = 2$ gauge theories.

The strong-coupling limit of SW theories corresponds to another limit of the integrable models, that of smooth finite-gap solutions degenerating into solitons [12]. Physically this is the weakly-coupled limit of a dual theory in which one expects monopole condensation, breaking symmetry down to $\mathcal{N} = 1$, and confinement [1]. As we shall see this is in precise correspondence with the properties of the solitonic solutions of the SW integrable systems, which correspond to particular values of the integrals of motions. The corresponding vacuum expectation values (VEV’s) are restricted to certain points in the moduli space, which can be treated as extrema of an $\mathcal{N} = 1$ superpotential.

The paper is organised as follows. In section 2 we begin with a discussion of the weak-coupling regime in $\mathcal{N} = 2$ SUSY gauge theories, presenting some intuitive motivation for why this can be directly related to the integrable systems of the Toda or Calogero-Moser families. Here we also review general properties of the corresponding finite-gap solutions. In section 3 we discuss in detail the perturbative limit, giving rise to the *open* Toda chain family. We present explicit formulas for the generating functions for the open Toda chain (Toda molecule) obtained along these line and discuss their relations with matrix models and duality in integrable systems. We discuss in a similar way the perturbative limit of elliptic Calogero-Moser models. In section 4 we turn to the solitonic or strong-coupling limit of the finite-gap integrable systems and derive the explicit form of solution in terms of the Baker-Akhiezer functions. This is the most physically attractive phase of the SW theories, corresponding to the confinement in $\mathcal{N} = 1$ gauge theories, and we show that the phase of the solitons is related to the string tensions in the confining phase. We also prove (in Appendix B) a conjecture of Edelman and Mas related to this strong coupling regime. We conclude with a discussion.

2 $\mathcal{N} = 2$ SUSY gauge theories and integrable systems

We begin in this section by reviewing various aspects of the correspondence between $\mathcal{N} = 2$ SUSY gauge theories and integrable systems that we shall need in our later discussion. Throughout we will limit our attention to those models arising in the context of either $D = 4$ pure $\mathcal{N} = 2$ $SU(N)$ gauge theories, or those with adjoint matter. These are listed in Figure 1. Extensions to other gauge groups and matter multiplets are possible, though will not be our focus here. Indeed, for concreteness we will describe the $N = 2$ cases of these models in some detail.

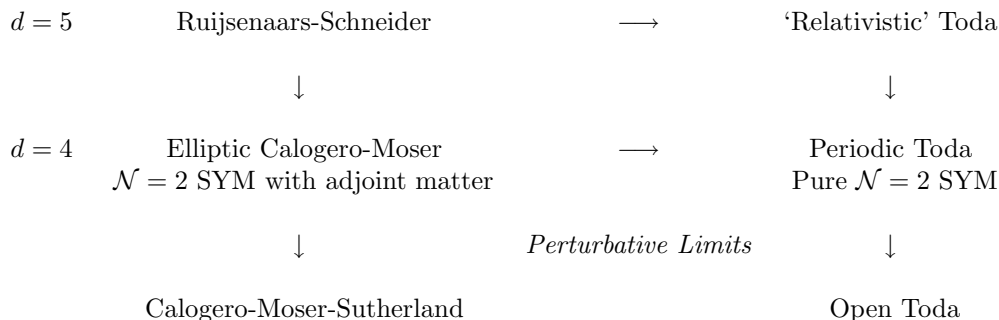


Figure 1.

We will begin the section with some intuitive motivation as to why the weak-coupling regime in $\mathcal{N} = 2$ SUSY gauge theories can be directly related to the integrable systems of the Toda or Calogero-Moser families. Then we review general properties of the corresponding finite-gap solutions and discuss in detail their perturbative limit, giving rise to the *open* Toda chain family.

Although much of what we say in this section is by way of review we will also present several new technical details concerning the relation between the Calogero-Moser and Ruijsenaars-Schneider spectral curves. The

books [7, 8] contain both recent reviews and general references for aspects of this correspondence not touched upon here.

2.1 Motivations: Perturbative Calculations and $N = 2$ Examples

The connections between SW theories and integrable systems can be discussed even at the *perturbative* level, where $\mathcal{N} = 2$ SUSY effective actions are completely determined by the 1-loop contributions. In the most well-known example, that of pure $\mathcal{N} = 2$ gauge theory with $SU(N)$ gauge group, the scalar field $\Phi = \|\Phi_{ij}\|$ of the $\mathcal{N} = 2$ vector supermultiplet may acquire nonzero VEV's $\Phi = \text{diag}(a_1, \dots, a_N)$ in extremising the potential $\text{Tr}[\Phi, \Phi^\dagger]^2$. This (generically) breaks the $SU(N)$ gauge group down to the maximal compact $U(1)^{N-1}$. The masses of W -bosons and their superpartners are (classically) given by $m_{ij}^W = a_{ij} \equiv a_i - a_j$ because of the interaction term $([A_\mu, \Phi]_{ij})^2 = (A_\mu^{ij}(a_i - a_j))^2$. By a simple technical trick these masses can be uniformly expressed in terms of the generating polynomial

$$w = P_N(\lambda) = \det_{N \times N}(\lambda - \Phi) = \prod_{i=1}^N (\lambda - a_i), \quad (1)$$

via the *contour integral*

$$m_{ij}^W = \oint_{C_{ij}} dS^{\text{pert}} = \oint_{C_{ij}} \lambda d \log w = \oint_{C_{ij}} \lambda d \log P_N(\lambda). \quad (2)$$

For a particular “figure-of-eight” like contour C_{ij} around the roots $\lambda = a_i$ and $\lambda = a_j$ this may be computed via the residue formula. This means that the contour integrals in the complex λ -plane (2) for such contours around these singular points (the roots of the polynomial (1)) give one set of the BPS masses in the SW theory. Another set of masses (monopoles) may be associated to *dual* contours starting and ending at the points a_i where dS^{pert} has singularities. Although such integrals are divergent, the result of such an integration says that the monopole masses are proportional to the masses m_{ij}^W , multiplied by the inverse square of the coupling. The divergences are absorbed by a renormalisation of the coupling. The integrals over the contours on the marked plane we are dealing with here are, as we shall later see, to be viewed in terms of contour integrals on a *degenerate* Riemann surface.

Now in $\mathcal{N} = 2$ perturbation theory the effective action (or prepotential) \mathcal{F} and the set of effective couplings T_{jk} , related by $-i\pi T_{jk} = \frac{\partial^2 \mathcal{F}}{\partial a_j \partial a_k}$, are determined by the 1-loop diagrams. This results in the logarithmic term

$$T_{jk}^{\text{pert}} = \frac{1}{2\pi i} \sum_{(\text{masses M})_{jk}} \log \frac{(\text{mass})^2}{\Lambda^2} = \frac{1}{2\pi i} \log \frac{a_{jk}^2}{\Lambda^2}, \quad (3)$$

where the scale parameter $\Lambda \equiv \Lambda_{QCD}$ so introduced may be related to the bare coupling τ . In the perturbative *weak-coupling* limit of the SW construction this is all one has. Instanton contributions to the prepotential which are proportional to powers of Λ^{2N} are suppressed, and one keeps only the terms proportional to $\log \Lambda$ or the bare coupling τ .

It remains to connect this discussion with integrable systems. The connection comes by interpreting the differential dS^{pert} with the generating differential of a Hamiltonian system. Remarkably, this interpretation extends beyond the perturbative regime! Lets consider the simplest SW theory [1], the $SU(2)$ pure gauge theory, where eq. (1) turns into

$$w = \lambda^2 - h. \quad (4)$$

Here $h = \frac{1}{2} \text{Tr} \Phi^2 = \frac{1}{2} a^2$ and the masses (2) are now defined by the contour integrals of

$$dS = \lambda d \log w = \frac{\lambda d\lambda}{\lambda - \sqrt{h}} + \frac{\lambda d\lambda}{\lambda + \sqrt{h}}. \quad (5)$$

Now, eqs. (4) and (5) can be interpreted as an *integration* of a simple dynamical system, the $SL(2)$ open Toda chain (or Liouville model) with Hamiltonian

$$H = \frac{1}{2}(p^2 + e^{2q}). \quad (6)$$

This has solution $e^q = a / \cosh(a[t - t_0])$, where $a = \sqrt{2H}$. One verifies that

$$\frac{p^2 dp}{p^2 - a^2} = pdq + \frac{apda}{p^2 - a^2} \quad (7)$$

and so upon integrating over a trajectory $\oint_{\gamma} \frac{p^2 dp}{p^2 - a^2} = \oint_{\gamma} p dq = 2\pi i a$. But with the co-ordinate $w = e^{2q}$, momentum $p = \lambda$ and Hamiltonian (energy) $H = h$ this is essentially (5). That is, integration of the canonical differential $dS = 2p dq$ over the trajectories of the Toda chain solutions gives rise, for various (complexified) trajectories, to the BPS masses in the SW theory.

This is actually a general rule. The perturbative mass spectrum and effective couplings of the $\mathcal{N} = 2$ theories of the “SW family” appear as actions when integrating the canonical differential of “open” or trigonometric families of integrable systems, the open Toda chain, the trigonometric Calogero-Moser and the trigonometric Ruijsenaars-Schneider systems. For pure gauge theories we have the N -particle (open) Toda chain with (rational) curves given by eq. (1) and generating differential given by (2). The inclusion of adjoint matter is associated to the trigonometric Calogero-Moser-Sutherland model where

$$w = \frac{P_N^{(CM)}(\lambda)}{P_N^{(CM)}(\lambda - m)} \quad dS = \lambda \frac{dw}{w}. \quad (8)$$

Five dimensional gauge theories whose $D = 4$ reductions have adjoint matter correspond to the trigonometric Ruijsenaars-Schneider system, where now

$$w = \frac{P_N^{(RS)}(\lambda)}{P_N^{(RS)}(\lambda e^{-2i\epsilon})} \quad dS = \log \lambda \frac{dw}{w}. \quad (9)$$

In each of these cases $P_N^{(CM)}$ and $P_N^{(RS)}$ are appropriate polynomials that we shall further describe in due course. It is easy to see that (the perturbative) spectra are given by the general formula [13]

$$M = a_{ij} \oplus \frac{\pi n}{R} \oplus \frac{\epsilon + \pi n}{R}, \quad n \in \mathbf{Z}. \quad (10)$$

In addition to the Higgs part a_{ij} this contains the Kaluza-Klein (KK) modes $\frac{\pi n}{R}$ and the KK modes for fields with “ ϵ -shifted” boundary conditions. The effective couplings are defined by almost the same formula as (3)

$$T_{ij}^{\text{pert}} = \frac{1}{2\pi i} \sum_{(\text{masses } M)_{ij}} \log \frac{M^2}{\Lambda^2} \quad (11)$$

i.e. by the sum of all the logarithms of spectrum (10) giving contribution for a particular 1-loop diagram for T_{ij} .

A priori nothing so simple can be said about the spectrum and structure of the theory at strong couplings. However the structure of the “strongly-coupled” phase is, as we will see below using the correspondence with integrable systems, similar in many respects to the weak-coupling regime, and may be described by explicitly computable expressions.

2.2 Generalities: finite-gap or Hitchin systems

Let us now consider the general setting. Our analysis so far has been built upon a (polynomial or rational) relation $R(\lambda, w) = 0$, a differential dS and BPS data given by integrating dS over various contours. We will now consider $R(\lambda, w) = 0$ as the equation describing the spectral curve of an integrable system. The differential, contours and symplectic form may be canonically described in terms of this integrable system. At heart will be the choice of integrable system. For the SW theories under discussion these are given by finite gap integrable systems of particles of a particular kind, the so-called Hitchin systems [14, 15]. We shall now describe some of these general features and give explicit descriptions of the Hitchin systems of relevance to us. The remarkable feature of the SW integrable systems correspondence is that non-perturbative aspects of the field theory are incorporated into this construction [5, 6].

Replacing the relation $R(\lambda, w) = 0$ of (1), (8) and (9) one now has the Lax equation of a spectral curve,

$$\det(\lambda - \mathcal{L}(z)) = 0. \quad (12)$$

Here the Lax operator $\mathcal{L}(z)$ is a matrix, defined on some *base* curve Σ_0 on which the spectral parameter z lies. For us this base curve is usually a torus (for elliptic models) or a sphere with punctures (for rational or trigonometric³ models). The two further ingredients were the generating differential dS and the contours C_{ij} .

³In perturbative examples it is parameterized by $w = e^z$.

The generating differential can now be defined by

$$\begin{aligned} dS &= \lambda dz \\ \delta_{\text{moduli}} dS &= \text{holomorphic differential.} \end{aligned} \quad (13)$$

The *action* variables are given by the Seiberg-Witten contour integrals over half of the independent contours

$$\mathbf{a} = \oint_{\mathbf{A}} dS, \quad (14)$$

or

$$\mathbf{a}^D = \oint_{\mathbf{B}} dS. \quad (15)$$

where $\mathbf{A} = \{A_1, \dots, A_g\}$ and $\mathbf{B} = \{B_1, \dots, B_g\}$ is a standard homology basis of $H^1(\Sigma)$.

The holomorphic variation in (13) is crucial to the definition of the symplectic form

$$\Omega = \delta dS|_{\gamma} = d\mathbf{a} \wedge d\mathbf{z}(\gamma) = d\mathbf{p} \wedge d\mathbf{q}. \quad (16)$$

The variation here is to be computed at the divisor $\gamma = \{\gamma_1, \dots, \gamma_g\}$ of the poles of the Baker-Akhiezer function (which is determined by $\mathcal{L}(z)$) which are co-ordinates on Σ^g . Let $\{d\omega_i\}$ be the set of *canonically* normalised holomorphic differentials $\oint_{A_i} d\omega_j = \delta_{ij}$. The variation of dS is understood as a total external derivative on $\text{Moduli}_g \ltimes \Sigma^g$,

$$\delta dS = \delta(\lambda dz) = \delta_{\text{moduli}} \lambda \wedge dz + d\lambda \wedge dz. \quad (17)$$

For this one must choose a *connection* on the bundle over moduli space [16] such that $\delta_{\text{moduli}} z = 0$, i.e. the parameter on the base curve z is covariantly constant. In practice this means that in the equation on two variables λ and z which defines the spectral curve Σ , we consider z as an independent variable, with the variable λ depending on moduli through eq. (12). Since on a one-dimensional curve Σ for any differentials one has $d\lambda \wedge dz = 0$, from (17) we finally get

$$\delta dS = \delta_{\text{moduli}} \lambda \wedge dz = \delta a_i \wedge \frac{\partial \lambda}{\partial a_i} dz = \delta a_i \wedge \frac{\partial dS}{\partial a_i} = \delta a_i \wedge d\omega_i. \quad (18)$$

Finally we introduce co-ordinates \mathbf{z} on the Jacobian by

$$z_i = \int_{\gamma_0}^{\gamma} d\omega_i + z_i^{(0)} \equiv \sum_{j=1}^g \int_{\gamma_0}^{\gamma_j} d\omega_i + z_i^{(0)}, \quad (19)$$

with $z_i^{(0)} = z_i^{(0)}(\gamma_0)$. Then

$$\Omega = \sum_{k=1}^g \delta dS(\gamma_k) = \delta a_i \wedge \sum_{k=1}^g d\omega_i(\gamma_k) \stackrel{(19)}{=} \delta a_i \wedge \delta z_i. \quad (20)$$

Our discussion thus far is general, predicated only upon a Lax operator and its attendant Baker-Akhiezer function. We must now describe the Lax operators for the class of models under consideration. These will be particular examples of Hitchin systems. For these the Lax operator in (12) $\mathcal{L}(z) \equiv \Phi$ is considered as a meromorphic matrix-valued function (or better, a 1-differential) on the base curve Σ_0 [11, 14, 15] satisfying

$$\bar{\partial}\Phi + [\bar{A}, \Phi] = \sum_{\alpha} J^{(\alpha)} \delta^{(2)}(P - P_{\alpha}). \quad (21)$$

The right hand side serve as sources, and $J^{(\alpha)}$ are matrices whose structure (given shortly for a single puncture) is particularly simple for the $su(n)$ theories. Thus Φ is *holomorphic* in the complex structure determined by \bar{A} on the punctured curve $\Sigma_0/\{P_{\alpha}\}$. The invariants of \bar{A} can be thought of as co-ordinates (one commuting set of variables) while the invariants of Φ as hamiltonians (another commuting set of variables) of an integrable system. The most general features of these “holomorphic” finite-gap [11] or Hitchin [14, 15] systems are:

- The spectral curve Σ_g covers some base curve Σ_0 (typically of genus $g_0 = 0, 1$) and the g moduli of the cover are viewed as distinguished with the moduli of the base curve viewed as “fixed” or “external” parameters”. This set of distinguished moduli are to be the constants of motion of our integrable system.

- In the general context of finite-gap integrable systems one may view this set-up in a different way. A generic complex curve depends on $3g - 3$ parameters describing the complex structure. The integrable system is described [11, 16] by two meromorphic differentials $d\lambda$ and dz on the curve. By adding holomorphic differentials these may be taken as having vanishing **A** periods. Fixing the **B** periods specifies the integrable system. Loosely, $d\lambda$ and dz are only defined up to multiples, and fixing the **B** periods of $d\lambda$ gives $g - 1$ constraints (taking account of the scaling freedom), while the **B** periods of dz give a further $g - 2$ constraints (now allowing $dz \rightarrow \alpha dz + \beta d\lambda$). Altogether we come to a system with $(3g - 3) - (g - 1) - (g - 2) = g$ parameters, the genus of Σ .
- This construction (when $g_0 = 0, 1$) implies certain linear relations on the period matrix T_{ij} :

$$\sum_j T_{ij} = \tau = \text{fixed.} \quad (22)$$

This comes from the possibility to choose the homology basis of Σ such that the **A** and **B** cycles “project” to the A and B cycles on the base torus Σ_0 . (A rational base curve can be thought of as degenerated torus.) Then

$$\sum_i a_i = \sum_i \oint_{A_i} \lambda dz = \oint_A \left(\sum \lambda_i \right) dz = \sum \mathcal{L}_{ii} \oint_A dz = h_1 \oint_A dz = h_1, \quad (23)$$

and

$$\sum_i a_i^D = \sum_i \oint_{B_i} \lambda dz = \oint_B \left(\sum \lambda_i \right) dz = \sum \mathcal{L}_{ii} \oint_B dz = h_1 \oint_B dz = h_1 \tau. \quad (24)$$

Thus, for any j ,

$$\sum_i T_{ij} = \sum_i \frac{\partial a_i^D}{\partial a_j} = \frac{\partial}{\partial a_j} \sum_i a_i^D \stackrel{(24)}{=} \frac{\partial h_1}{\partial a_j} \tau \stackrel{(23)}{=} \tau \quad (25)$$

giving (22).

These properties will be crucial in our discussion of the *tau-functions* (basically the Riemann theta-functions on the corresponding spectral curves Σ) for these finite-gap integrable systems.

Let us now give three particular examples of the Hitchin integrable models arising in the context of SW theory: the Toda chain, the elliptic Calogero-Moser model and the Ruijsenaars-Schneider model. These are integrable systems we encountered in the perturbative discussion earlier.

Toda chain. The Toda chain system [17] is a system of N particles with nearest neighbour exponential interactions:

$$\frac{\partial q_i}{\partial t} = p_i \quad \frac{\partial p_i}{\partial t} = e^{q_{i+1} - q_i} - e^{q_i - q_{i-1}}. \quad (26)$$

It is an integrable system, with N Poisson-commuting Hamiltonians, $h_1 = \sum p_i = P$, $h_2 = \sum \left(\frac{1}{2} p_i^2 + e^{q_i - q_{i-1}} \right) = E$, etc. We may have either an open chain ($q_0 = -\infty, q_{N+1} = \infty$) or a periodic system (of “period” N : $q_{i+N} = q_i$ and $p_{i+N} = p_i$). The periodic problem can be derived by reduction of the infinite-dimensional system of particles on the line by aid of two commuting operators: the Lax operator \mathcal{L} (for the auxiliary linear problem for (26))

$$\lambda \Psi_n = \sum_k \mathcal{L}_{nk} \Psi_k = e^{\frac{1}{2}(q_{n+1} - q_n)} \Psi_{n+1} + p_n \Psi_n + e^{\frac{1}{2}(q_n - q_{n-1})} \Psi_{n-1} \quad (27)$$

and a second operator, the *monodromy* or shift operator in the discrete variable (the particle number)

$$T q_n = q_{n+N} \quad T p_n = p_{n+N} \quad T \Psi_n = \Psi_{n+N}. \quad (28)$$

The existence of a common spectrum for these two operators

$$\mathcal{L} \Psi = \lambda \Psi \quad T \Psi = w \Psi \quad [\mathcal{L}, T] = 0 \quad (29)$$

means that there is a relation between them $\mathcal{P}(\mathcal{L}, T) = 0$. This in fact is the equation of the spectral curve Σ : $\mathcal{P}(\lambda, w) = 0$.

One way to get explicit form of the spectral curve equation is to rewrite the Lax operator (27) in the basis of the T -operator eigenfunctions.⁴ This then becomes a $N \times N$ matrix,

$$\mathcal{L} = \mathcal{L}^{TC}(w) = \mathbf{p} \cdot \mathbf{H} + \sum_{\text{simple } \alpha} e^{\alpha \cdot \mathbf{q}} (E_{\alpha} + E_{-\alpha}) + w^{-1} e^{-\alpha_0 \cdot \mathbf{q}} E_{-\alpha_0} + w e^{-\alpha_0 \cdot \mathbf{q}} E_{\alpha_0}$$

$$= \begin{pmatrix} p_1 & e^{\frac{1}{2}(q_2 - q_1)} & 0 & \dots & w e^{\frac{1}{2}(q_1 - q_N)} \\ e^{\frac{1}{2}(q_2 - q_1)} & p_2 & e^{\frac{1}{2}(q_3 - q_2)} & \dots & 0 \\ 0 & e^{\frac{1}{2}(q_3 - q_2)} & p_3 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \frac{1}{w} e^{\frac{1}{2}(q_1 - q_N)} & 0 & 0 & \dots & p_N \end{pmatrix}. \quad (30)$$

Here the sum is over the simple roots, and $\alpha_0 = -\sum_{\text{simple } \alpha} \alpha$ is minus the highest root. The construction depends explicitly on the eigenvalue w of the shift operator (28). This is the spectral parameter which is defined on the cylinder $\Sigma_0 = \mathbf{CP}^1 \setminus \{0, \infty\}$. The eigenvalues of the Lax operator (30) are defined from the spectral equation

$$\mathcal{P}(\lambda, w) = \det_{N \times N} (\lambda - \mathcal{L}^{TC}(w)) = 0. \quad (31)$$

Substituting the explicit expression (30) into (31), one obtains

$$\mathcal{P}(\lambda, w) = P_N(\lambda) - w - \frac{1}{w} = 0, \quad P_N(\lambda) = \lambda^N + \sum_{k=1}^N (-1)^k h_k \lambda^{N-k}, \quad (32)$$

which for $N = 2$ is

$$w + \frac{1}{w} = \lambda^2 - (p_1 + p_2)\lambda + p_1 p_2 - (e^{q_2 - q_1} + e^{q_1 - q_2}). \quad (33)$$

The generating 1-form $dS = \lambda \frac{dw}{w}$ indeed satisfies (13)

$$\delta_{\text{moduli}} dS \equiv \delta_{\text{moduli}} dS|_{w=\text{const}} = (\delta_{\text{moduli}} \lambda) \frac{dw}{w} = \frac{\sum \lambda^k \delta h_k}{P'_N(\lambda)} \frac{dw}{w} = \sum \frac{\lambda^k d\lambda}{y} \delta h_k = \text{holomorphic}. \quad (34)$$

where

$$y^2 = \left(w - \frac{1}{w}\right)^2 = P_N^2(\lambda) - 4, \quad P'_N(\lambda) \equiv \left. \frac{\partial P_N}{\partial \lambda} \right|_{h_k=\text{const}} \quad (35)$$

By the gauge transformation $U_{ij} = v^i \delta_{ij}$ with $w \equiv v^N$ the Lax operator (30) can be brought to another familiar form

$$\begin{aligned} \mathcal{L}^{TC}(w) &\rightarrow \tilde{\mathcal{L}}^{TC}(v) = U^{-1} \mathcal{L}^{TC}(w) U \\ &= \mathbf{p} \cdot \mathbf{H} + v^{-1} \left(e^{-\alpha_0 \cdot \mathbf{q}} E_{-\alpha_0} + \sum_{\text{simple } \alpha} e^{\alpha \cdot \mathbf{q}} E_{\alpha} \right) + v \left(e^{-\alpha_0 \cdot \mathbf{q}} E_{\alpha_0} + \sum_{\text{simple } \alpha} e^{\alpha \cdot \mathbf{q}} E_{-\alpha} \right) \\ &= \begin{pmatrix} p_1 & \frac{1}{v} e^{\frac{1}{2}(q_2 - q_1)} & 0 & \dots & v e^{\frac{1}{2}(q_1 - q_N)} \\ v e^{\frac{1}{2}(q_2 - q_1)} & p_2 & \frac{1}{v} e^{\frac{1}{2}(q_3 - q_2)} & \dots & 0 \\ 0 & v e^{\frac{1}{2}(q_3 - q_2)} & p_3 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \frac{1}{v} e^{\frac{1}{2}(q_1 - q_N)} & 0 & 0 & \dots & p_N \end{pmatrix}. \end{aligned} \quad (36)$$

In this form [18] it clearly satisfies the $\bar{\partial}$ -equation (21) on a cylinder with *trivial* gauge connection

$$\begin{aligned} \bar{\partial}_v \tilde{\mathcal{L}}^{TC}(v) &= \frac{\partial}{\partial \bar{v}} \tilde{\mathcal{L}}^{TC}(v) = \left(e^{-\alpha_0 \cdot \mathbf{q}} E_{-\alpha_0} + \sum_{\text{simple } \alpha} e^{\alpha \cdot \mathbf{q}} E_{\alpha} \right) \delta(P_0) \\ &\quad - \left(e^{-\alpha_0 \cdot \mathbf{q}} E_{\alpha_0} + \sum_{\text{simple } \alpha} e^{\alpha \cdot \mathbf{q}} E_{-\alpha} \right) \delta(P_{\infty}). \end{aligned} \quad (37)$$

⁴If we had chosen to work instead with \mathcal{L} , which is a *second-order* difference operator, we would come to the Faddeev-Takhtajan 2×2 formalism of Toda chains.

The Toda chain coupling Λ (equal to the mass scale Λ_{QCD} in the SW theory) can be restored by rescaling $\lambda \rightarrow \lambda/\Lambda$, $w \rightarrow w/\Lambda^N$ and the hamiltonians $h_k \rightarrow \frac{h_k}{\Lambda^k}$. Then equation (32) acquires the form of

$$w + \frac{\Lambda^{2N}}{w} = P_N(\lambda) = \lambda^N + \sum_{k=1}^N (-1)^k h_k \lambda^{N-k} \quad (38)$$

proposed in the context of SW theory [2].

Elliptic Calogero-Moser model. The $N \times N$ matrix Lax operator (36) (and, therefore, (30)) can be thought of as a “degenerate” case of the Lax operator for the N -particle Calogero-Moser system [19]

$$\begin{aligned} \mathcal{L}^{CM}(z) &= \mathbf{p} \cdot \mathbf{H} + \sum_{\alpha} F(\mathbf{q} \cdot \alpha | z) E_{\alpha} \\ &= \begin{pmatrix} p_1 & F(q_1 - q_2 | z) & \dots & F(q_1 - q_N | z) \\ F(q_2 - q_1 | z) & p_2 & \dots & F(q_2 - q_N | z) \\ \vdots & & \ddots & \vdots \\ F(q_N - q_1 | z) & F(q_N - q_2 | z) & \dots & p_N \end{pmatrix}. \end{aligned} \quad (39)$$

The sum here is now over all of the roots and the base curve Σ_0 is a torus instead of a cylinder. The matrix elements

$$F(q | z) = im \frac{\sigma(q + z)}{\sigma(q)\sigma(z)} \quad (40)$$

are defined in terms of the Weierstrass sigma-functions $\sigma(z)$. Equivalently $\sigma(z) = 2\omega e^{\frac{\eta z^2}{2\omega} \frac{\theta_*(z)}{\theta'_*(0)}}$ where $\theta_*(z) \equiv \theta[\frac{1}{1}](z) \equiv \theta_1(z)$ is the (only) odd Jacobi theta-function. The modulus of the elliptic curve Σ_0 is τ and we have the marked point $z = 0$ at which $x = \wp(z) = \infty$, $y = \frac{1}{2}\wp'(z) = \infty$. The canonical holomorphic 1-differential on Σ_0 is $dz = \frac{dx}{2y}$. The Lax operator (39) corresponds to a completely integrable system with Hamiltonians $h_1 = \sum_i p_i$, $h_2 = \sum_{i < j} (p_i p_j - m^2 \wp(q_i - q_j))$, $h_3 = \sum_{i < j < k} (p_i p_j p_k + \dots)$, etc.

From (12), (39) it follows that the spectral curve Σ^{CM} for the N -particle Calogero-Moser system

$$\det_{N \times N} (\lambda - \mathcal{L}^{CM}(z)) = 0 \quad (41)$$

covers N times the base elliptic curve. These spectral curves are very special: in general the genus of a curve defined by an $N \times N$ matrix grows as N^2 while the curve (41) has genus $g = N$. For $N = 2$ we have

$$0 = \lambda^2 - (p_1 + p_2)\lambda + p_1 p_2 - m^2 (\wp(q_2 - q_1) - \wp(z)). \quad (42)$$

The BPS masses \mathbf{a} and \mathbf{a}^D are now related to the periods of generating 1-differential

$$dS^{CM} = 2\lambda dz = \lambda \frac{dx}{y} \quad (43)$$

along the non-contractable contours on Σ^{CM} .

In order to recover the Toda-chain system, one has to take the double-scaling limit [20], when m and $-i\tau$ both go to infinity and

$$q_i - q_j \rightarrow \frac{1}{2}(i - j) \log m + (q_i - q_j) \quad (44)$$

so that the dimensionless coupling τ gets substituted by a dimensionful parameter $\Lambda^N \sim m^N e^{i\pi\tau}$. The idea is to separate the pairwise interacting particles far away from each other and to adjust the coupling constant simultaneously in such a way, that only the interaction of nearest particles survives (and turns in an exponential). This limit is described in more detail in Appendix A. In this limit, the elliptic curve degenerates into a cylinder with coordinate $w = e^z e^{i\pi\tau}$ so that $dz \rightarrow \frac{dw}{w}$ and

$$dS^{CM} \rightarrow dS^{TC} = \lambda \frac{dw}{w}. \quad (45)$$

The Lax operator of the Calogero system turns into that of the N -periodic Toda chain (30):

$$\mathcal{L}^{CM}(z) dz \rightarrow \mathcal{L}^{TC}(w) \frac{dw}{w} \quad (46)$$

and the spectral curve acquires the form (31).

One further remark is in order concerning the dependence of the elliptic Calogero-Moser model on the coupling constant, or equivalently the mass of the adjoint hypermultiplet m . The exact equivalence between the Calogero-Moser and KP theories is usually considered only when $m^2 = 2$ (or at least $m^2 = n(n+1)$ for integer n). This restriction is however only essential when we consider the “Lax” equation for the “first” KP time, the x -variable,

$$\left(-\frac{\partial^2}{\partial x^2} + m^2 \sum \wp(x - q_i)\right) \Psi = E\Psi. \quad (47)$$

This has solution with simple poles $\Psi \sim \sum \frac{c_i}{x - q_i} + \dots$ only when $m^2 = 2$ (the condition of cancellation of the highest pole). However, for the Calogero-Moser equations themselves ($\frac{d^2 q_i}{dt^2} = m^2 \sum_{i < j} \wp'(q_{ij})$ and the “higher” hamiltonian equations) there is no such restriction on the coupling constant. One can always set $m^2 = 2$ by a simple rescaling of the time-variables $t \rightarrow mt$. The point is that in achieving the canonical forms of the KP and KdV equations all of the possible scalings have already been employed: consequently the coupling constant is restricted.

Calogero-Moser spectral curve from the elliptic Ruijsenaars. The easiest and most general way to look at Seiberg-Witten theories with adjoint matter is to consider the elliptic Ruijsenaars-Schneider model [13]. The Lax operator of the elliptic Ruijsenaars model has the form [21]

$$\mathcal{L}_{ij}^R = e^{P_i} \frac{\sigma(q_{ij} + z + \epsilon)\sigma(\epsilon)}{\sigma(q_{ij} + \epsilon)\sigma(z + \epsilon)}, \quad e^{P_i} = e^{p_i} \prod_{k \neq i} \sigma(\epsilon) \sqrt{\wp(\epsilon) - \wp(q_{ik})}. \quad (48)$$

In the trigonometric limit this turns into

$$\mathcal{L}_{ij}^{TR} = e^{P_i} \frac{\sinh(q_{ij} + z + \epsilon) \sinh(\epsilon)}{\sinh(q_{ij} + \epsilon) \sinh(z + \epsilon)}, \quad e^{P_i} = e^{p_i} \prod_{k \neq i} \sqrt{1 - \frac{\sinh^2 \epsilon}{\sinh^2(q_{ik})}}. \quad (49)$$

Introducing $\nu_i = e^{2q_i}$, $\zeta = e^{2z}$ and $q = e^{2\epsilon}$ one finds that

$$\mathcal{L}_{ij}^{TR} = e^{P_i} \frac{q\zeta\nu_i - \nu_j}{q\nu_i - \nu_j} \frac{q - 1}{q\zeta - 1} \xrightarrow{\zeta \rightarrow \infty} (q - 1) \frac{e^{P_i} \nu_i}{q\nu_i - \nu_j} + \mathcal{O}\left(\frac{1}{\zeta}\right), \quad e^{P_i} = e^{p_i} \prod_{k \neq i} \frac{\sqrt{q\nu_i - \nu_j} \sqrt{\nu_i - q\nu_j}}{\nu_i - \nu_j}. \quad (50)$$

Often only the leading term in (50) is taken as an expression for the Lax operator of the trigonometric Ruijsenaars system.

Using (48) the spectral curve equation for the elliptic Ruijsenaars can be written as ⁵

$$\det_{N \times N} (\lambda - \mathcal{L}^R(z)) = \sum_{k=0}^N \lambda^{N-k} (-)^k \sum_{|I|=k} \prod_{i \in I} e^{P_i} \det_{k \times k} \frac{\sigma(q_{ij} + z + \epsilon)\sigma(\epsilon)}{\sigma(q_{ij} + \epsilon)\sigma(z + \epsilon)} \Big|_{i,j \in I} \quad (51)$$

where, using Wick’s theorem for the correlation functions of free fermions [22, 21, 13], for the determinants one gets

$$\begin{aligned} \det_{k \times k} \frac{\sigma(q_{ij} + z + \epsilon)\sigma(\epsilon)}{\sigma(q_{ij} + \epsilon)\sigma(z + \epsilon)} &= \det_{k \times k} \frac{\theta_*(q_{ij} + z + \epsilon)\theta_*(\epsilon)}{\theta_*(q_{ij} + \epsilon)\theta_*(z + \epsilon)} \Big|_{\tilde{q}_j = q_j - \epsilon} = \det_{k \times k} \frac{\theta_*(q_i - \tilde{q}_j + z)}{\theta_*(q_i - \tilde{q}_j)\theta_*(z)} \frac{\theta_*(z)\theta_*(\epsilon)}{\theta_*(z + \epsilon)} \\ &= \left(\frac{\theta_*(z)\theta_*(\epsilon)}{\theta_*(z + \epsilon)} \right)^k \det_{k \times k} \langle \psi(q_i) \tilde{\psi}(q_j) \rangle_z = \left(\frac{\theta_*(z)\theta_*(\epsilon)}{\theta_*(z + \epsilon)} \right)^k \langle \prod_{i \in I} \psi(q_i) \prod_{j \in I} \tilde{\psi}(q_j) \rangle_z \\ &= \left(\frac{\theta_*(z)\theta_*(\epsilon)}{\theta_*(z + \epsilon)} \right)^k \frac{\prod_{i < j} \theta_*(q_i - q_j) \prod_{i < j} \theta_*(\tilde{q}_i - \tilde{q}_j)}{\prod_{i,j} \theta_*(q_i - \tilde{q}_j)} \frac{\theta_*(\sum q_i - \sum \tilde{q}_i + z)}{\theta_*(z)} \\ &= \left(\frac{\theta_*(z)}{\theta_*(z + \epsilon)} \right)^k \frac{\prod_{i < j} \theta_*(q_{ij})^2}{\prod_{i \neq j} \theta_*(q_{ij} + \epsilon)} \frac{\theta_*(z + k\epsilon)}{\theta_*(z)}. \end{aligned} \quad (52)$$

Using (52) and introducing $\tilde{\lambda} \equiv \lambda \frac{\theta_*(z + \epsilon)}{\theta_*(z)}$, one finally gets for (51)

$$\det_{N \times N} (\lambda - \mathcal{L}^R(z)) = \left(\frac{\theta_*(z)}{\theta_*(z + \epsilon)} \right)^N \left[\sum_{k=0}^N \tilde{\lambda}^{N-k} h_k \frac{\theta_*(z + k\epsilon)}{\theta_*(z)} \right] = 0 \quad (53)$$

where the hamiltonians h_k are

$$\begin{aligned} (-)^k h_k &\equiv \sum_{|I|=k} \prod_{i \in I} e^{P_i} \frac{\prod_{i < j \in I} \theta_*(q_{ij})^2}{\prod_{i \neq j \in I} \theta_*(q_{ij} + \epsilon)} = \sum_{|I|=k} \prod_{i \in I} e^{P_i} \prod_{i < j \in I} \frac{\theta_*(q_{ij})^2}{\theta_*(q_{ij} + \epsilon)\theta_*(-q_{ij} + \epsilon)} \\ &= \sum_{|I|=k} \prod_{i \in I} e^{P_i} \frac{1}{\prod_{i < j \in I} \theta_*(\epsilon)^2} \frac{1}{\wp(\epsilon) - \wp(q_{ij})} \end{aligned} \quad (54)$$

⁵This is a particular case of generic determinant formulas (147), considered in more detail below.

where we used

$$\frac{\theta_*(q+\epsilon)\theta_*(q-\epsilon)}{\theta_*(q)^2\theta_*(\epsilon)^2} = \frac{\sigma(q+\epsilon)\sigma(q-\epsilon)}{\sigma(q)^2\sigma(\epsilon)^2} = \wp(\epsilon) - \wp(q).$$

For $N = 2$ we obtain

$$\begin{aligned} \det_{2 \times 2} (\lambda - \mathcal{L}^R(z)) &= \lambda^2 - \lambda (e^{P_1} + e^{P_2}) + e^{P_1+P_2} \frac{\wp(\epsilon) - \wp(z+\epsilon)}{\wp(\epsilon) - \wp(q_{12})} \\ &= \lambda^2 - \lambda \sigma(\epsilon) (e^{P_1} + e^{P_2}) \sqrt{\wp(\epsilon) - \wp(q_{12})} + e^{P_1+P_2} \sigma^2(\epsilon) (\wp(\epsilon) - \wp(z+\epsilon)). \end{aligned} \quad (55)$$

The spectral curve of the elliptic Calogero-Moser model (39) arises in the $\epsilon \rightarrow 0$ limit of (53). Indeed, in the limit $\epsilon = mR$, $p_i \rightarrow Rp_i$ and $R \rightarrow 0$, one recovers from (48)

$$\mathcal{L}_{ii}^R = e^{P_i} = e^{Rp_i} \prod_{k \neq i} \sigma(mR) \sqrt{\wp(mR) - \wp(q_{ik})} = 1 + Rp_i + \mathcal{O}(R^2), \quad (56)$$

$$\mathcal{L}_{ij}^R = e^{Rp_i} \prod_{k \neq i} \sigma(mR) \sqrt{\wp(mR) - \wp(q_{ik})} \frac{\sigma(q_{ij} + z + mR) \sigma(mR)}{\sigma(q_{ij} + mR) \sigma(z + mR)} = R \cdot m \frac{\sigma(q_{ij} + z)}{\sigma(q_{ij}) \sigma(z)} + \mathcal{O}(R^2) \quad i \neq j.$$

Thus

$$\mathcal{L}_{ij}^R = \delta_{ij} + R \left(p_i \delta_{ij} + m(1 - \delta_{ij}) \frac{\sigma(q_{ij} + z)}{\sigma(q_{ij}) \sigma(z)} \right) + \mathcal{O}(R^2) = \delta_{ij} + R \mathcal{L}_{ij}^{CM} + \mathcal{O}(R^2) \quad (57)$$

and one obtains

$$\begin{aligned} \det_{N \times N} (\lambda - \mathcal{L}^R(z)) &= \det_{N \times N} (\lambda - 1 - R \mathcal{L}^{CM}(z) + \mathcal{O}(R^2)) \\ &= R^N \det_{N \times N} \left(\frac{\lambda-1}{R} - \mathcal{L}^{CM}(z) \right) + \mathcal{O}(R^{N+1}). \end{aligned} \quad (58)$$

Provided the Ruijsenaars parameter scales as $\frac{\lambda-1}{R} \rightarrow \lambda^{CM}$ then in the limit $R \rightarrow 0$ we obtain exactly the spectral equation (41).

Lets consider these for the $N = 2$ example. In this case one has from (55)

$$\begin{aligned} \det_{2 \times 2} (\lambda - \mathcal{L}^R(z)) &= (\lambda - 1)^2 - mR(\lambda - 1)(p_1 + p_2) \\ &\quad + (mR)^2 \left(\lambda \wp(q_{12}) - \frac{1}{2} \lambda (p_1^2 + p_2^2) + \frac{1}{2} (p_1 + p_2)^2 - \wp(z) \right) + \mathcal{O}(R^3) \end{aligned} \quad (59)$$

After the substitutions $\lambda - 1 \rightarrow R\lambda'$, one finally gets from (59)

$$\det_{2 \times 2} (R\lambda' + 1 - \mathcal{L}^R(z)) \underset{R \rightarrow 0}{=} R^2 (\lambda'^2 - \lambda' (p_1 + p_2) + p_1 p_2 + m^2 \wp(q_{12}) - m^2 \wp(z)) + \mathcal{O}(R^3) \quad (60)$$

which is the $N = 2$ elliptic Calogero-Moser curve up to $R^{N+1} = R^3$ terms. On the other hand the $N = 2$ spectral curve (59) can be rewritten in the form (53)

$$\det_{2 \times 2} (\lambda - \mathcal{L}^R(z)) = \left(\frac{\sigma(z)}{\sigma(z+\epsilon)} \right)^2 \left(\tilde{\lambda}^2 - \tilde{\lambda} \frac{\sigma(z+\epsilon)}{\sigma(z)} h_1^R + \frac{\sigma(z+2\epsilon)}{\sigma(z)} e^{p_1+p_2} \right) \quad (61)$$

where

$$\tilde{\lambda} = \lambda \frac{\sigma(z+\epsilon)}{\sigma(z)}, \quad h_1^R = (e^{P_1} + e^{P_2}) \sigma(\epsilon) \sqrt{\wp(\epsilon) - \wp(q_{12})}. \quad (62)$$

With $p_i \rightarrow \epsilon p_i$ then in the limit $\epsilon = mR \rightarrow 0$ this may be expressed as

$$\begin{aligned} \tilde{\lambda}^2 - \tilde{\lambda} \frac{\sigma(z+\epsilon)}{\sigma(z)} h_1^R + \frac{\sigma(z+2\epsilon)}{\sigma(z)} e^{p_1+p_2} &= (\tilde{\lambda} - 1)^2 - \epsilon (\tilde{\lambda} - 1) (h_1^{CM} + 2 \frac{\sigma'(z)}{\sigma(z)}) \\ &\quad + \epsilon^2 \left(\tilde{\lambda} h_2^{CM} + (2 - \tilde{\lambda}) \left(\frac{\sigma''(z)}{\sigma(z)} - \frac{\sigma'(z)}{\sigma(z)} h_1^{CM} \right) + \frac{1}{2} (\tilde{\lambda} - 1) (h_1^{CM})^2 \right) + \mathcal{O}(\epsilon^3). \end{aligned} \quad (63)$$

Here we have used that $h_1^R = 2 + \epsilon h_1^{CM} + \epsilon^2 (\frac{1}{2} (h_1^{CM})^2 - h_2^{CM})$. Finally after the substitution $\tilde{\lambda} - 1 \rightarrow \epsilon \lambda'$, one arrives at

$$\begin{aligned} \tilde{\lambda}^2 - \tilde{\lambda} \frac{\sigma(z+\epsilon)}{\sigma(z)} h_1^R + \frac{\sigma(z+2\epsilon)}{\sigma(z)} e^{p_1+p_2} &= \epsilon^2 \left(\lambda'^2 - \lambda' h_1^{CM} + h_2^{CM} + (h_1^{CM} - 2\lambda') \frac{\sigma'(z)}{\sigma(z)} + \frac{\sigma''(z)}{\sigma(z)} \right) + \mathcal{O}(\epsilon^3) \\ &= \epsilon^2 \frac{\sigma(z - \frac{\partial}{\partial \lambda'})}{\sigma(z)} (\lambda'^2 - \lambda' h_1^{CM} + h_2^{CM}) + \mathcal{O}(\epsilon^3). \end{aligned} \quad (64)$$

One can also rewrite (64) as

$$\begin{aligned} \lambda'^2 &- \lambda' h_1^{CM} + h_2^{CM} + (h_1^{CM} - 2\lambda') \frac{\sigma'(z)}{\sigma(z)} + \frac{\sigma''(z)}{\sigma(z)} \\ &= \left(\lambda' - \frac{\sigma'(z)}{\sigma(z)} \right)^2 - \left(\lambda' - \frac{\sigma'(z)}{\sigma(z)} \right) h_1^{CM} - \frac{\sigma'(z)^2}{\sigma(z)^2} + \frac{\sigma''(z)}{\sigma(z)} - h_2^{CM} \\ &= \lambda^2 - \lambda h_1^{CM} + h_2^{CM} - \wp(z) \end{aligned} \quad (65)$$

which is a common representation for the $N = 2$ Calogero curve (see, for example formula (108) below, with $\lambda = \lambda' - \frac{\sigma'(z)}{\sigma(z)}$, $x = \wp(z)$ and $h_2^{CM} = -h$).

Returning to the general setting, instead of (64) one gets

$$\sum_{k=0}^N \tilde{\lambda}^{N-k} h_k \frac{\theta_*(z+k\epsilon)}{\theta_*(z)} \rightarrow \epsilon^N \frac{\theta_*(z - \frac{\partial}{\partial \lambda'})}{\theta_*(z)} \sum_{k=0}^N \lambda'^{N-k} h_k^{CM} + \mathcal{O}(\epsilon^{N+1}) \quad (66)$$

which is the D'Hoker-Phong form of the Calogero curve [24] (see also [25])⁶

$$\frac{\theta_*(z - \frac{\partial}{\partial \lambda'})}{\theta_*(z)} \sum_{k=0}^N \lambda'^{N-k} h_k^{CM} = \frac{\theta_*(z - \frac{\partial}{\partial \lambda'})}{\theta_*(z)} P_N(\lambda') = 0. \quad (68)$$

The Calogero-Moser-KP “quasimomentum” (that is the meromorphic differential with single first order pole and vanishing **A** periods) is

$$dp = d\lambda' = d\left(\lambda - \frac{\theta'_*(z)}{\theta_*(z)}\right) \quad (69)$$

and the Seiberg-Witten periods are

$$a_i = \oint_{A_i} \log \lambda dz, \quad a_i^D = \oint_{B_i} \log \lambda dz, \quad (70)$$

in the Ruijsenaars model. The period matrix is, as always

$$T_{ij} = \frac{\partial a_i^D}{\partial a_j}, \quad i, j = 1, \dots, N. \quad (71)$$

2.3 Theta Functions for the Calogero-Moser Family

We will next consider several simplifications that arise when the spectral curve (12) covers a 1-dimensional complex torus, or its degenerations, a cylinder or sphere with two punctures. This setting contains the elliptic Calogero-Moser and Ruijsenaars-Schneider models, corresponding to SW theories with adjoint matter, with the degenerations including the periodic Toda chain corresponding to pure gluodynamics. In the case of a curve covering a torus the theta functions defined on the Jacobian of the curve simplify, and these are the expressions we shall need.

The basic Riemann theta-function with characteristics is defined by

$$\Theta \left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix} \right] (\mathbf{z}|T) = \sum_{\mathbf{n} \in \mathbb{Z}^N} e^{2\pi i \sum_{i=1}^N (n_i + \frac{\epsilon_i}{2})(z_i + \frac{\epsilon'_i}{2}) + i\pi \sum_{i,j=1}^N (n_i + \frac{\epsilon_i}{2}) T_{ij} (n_j + \frac{\epsilon_j}{2})} \quad (72)$$

and we set $\Theta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] (\mathbf{z}|T) = \Theta(\mathbf{z}|T)$. By taking T to be the period matrix of a curve of genus $g = N$ we associate a theta function to the Jacobian of the curve. The period matrices of our models satisfy the constraint (22)

$$\sum_{i=1}^N T_{ij} \stackrel{\forall j}{=} \tau \stackrel{\forall i}{=} \sum_{j=1}^N T_{ij}, \quad (73)$$

and this allows simplifications. We will show that

$$\Theta(\mathbf{z}|T) = \sum_{k=0}^{N-1} \theta_{\frac{k}{N}}(z|N\tau) \Theta_k(\hat{\mathbf{z}}|\hat{T}) \quad (74)$$

⁶In (64) we have chosen σ -functions for convenience instead of θ_* -functions, this differs slightly from the general formula (68). However, one may easily check that this difference is inessential and the result is still

$$\epsilon^2 \frac{\theta_*(z - \frac{\partial}{\partial \lambda'})}{\theta_*(z)} \left(\lambda'^2 - \lambda' h_1^{CM} + h_2^{CM} + 2\eta \right) + \mathcal{O}(\epsilon^3) = \epsilon^2 \left(\lambda^2 - \lambda h_1^{CM} + h_2^{CM} - \wp(z) \right) + \mathcal{O}(\epsilon^3). \quad (67)$$

where $\theta_{\frac{k}{N}}(z|N\tau) \equiv \theta \left[\begin{smallmatrix} \frac{2k}{N} \\ 0 \end{smallmatrix} \right] (z|N\tau)$ is a genus $g = 1$ or Jacobi theta-function with characteristic, Θ_k is a genus $N - 1$ theta function and $z, \hat{\mathbf{z}}$ and \hat{T} will be defined shortly.

Let $\mathbf{e}_N = (1, \dots, 1)$ be the N vector with all 1's, and from this construct the projection matrix $P = \frac{1}{N} \mathbf{e}_N^T \mathbf{e}_N$. Because of (73) one may write

$$T = \tau P + \tilde{T}, \quad \tilde{T} = (1 - P)T(1 - P). \quad (75)$$

Thus $T_{ij} = \frac{\tau}{N} + \tilde{T}_{ij}$ and $\sum_{i=1}^N \tilde{T}_{ij} \stackrel{\forall j}{=} 0 \stackrel{\forall i}{=} \sum_{j=1}^N \tilde{T}_{ij}$. We may similarly decompose \mathbf{z} ,

$$\mathbf{z} = \frac{z}{N} \mathbf{e}_N + \tilde{\mathbf{z}}, \quad \tilde{\mathbf{z}} = (1 - P)\mathbf{z}. \quad (76)$$

Thus $z_i = \frac{z}{N} + \tilde{z}_i$ and $\sum_{i=1}^N z_i = z$ or $\sum_{i=1}^N \tilde{z}_i = 0$. In order to express $\Theta(\mathbf{z}|T)$ in the form (74) introduce the matrices

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & -1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix} \quad M^{-1} = \frac{1}{N} \begin{pmatrix} N-1 & -1 & \dots & -1 & 1 \\ -1 & N-1 & \dots & -1 & 1 \\ \vdots & & \ddots & & \vdots \\ -1 & -1 & \dots & N-1 & 1 \\ -1 & -1 & \dots & -1 & 1 \end{pmatrix} \quad (77)$$

and the change of basis

$$\begin{pmatrix} \hat{\mathbf{z}} \\ z \end{pmatrix} = M\mathbf{z}, \quad (78)$$

where $\hat{\mathbf{z}}$ is now an $N - 1$ vector. Then

$$\mathbf{n} \cdot \mathbf{z} = \mathbf{n}^T M^{-1} M \mathbf{z} = \sum_{j=1}^{N-1} (n_j - \frac{k}{N}) \hat{z}_j + \frac{z}{N} = (\hat{\mathbf{n}} - \frac{k}{N} \mathbf{e}_{N-1}) \cdot \hat{\mathbf{z}} + \frac{z}{N} \quad (79)$$

with $k = \sum_{j=1}^N n_j$. Also

$$\mathbf{n}^T T \mathbf{n} = \mathbf{n}^T M^{-1} M (\tau P + \tilde{T}) M^T M^{T-1} \mathbf{n} = \frac{\tau}{N} k^2 + \mathbf{n}^T M^{-1} M \tilde{T} M^T M^{T-1} \mathbf{n}. \quad (80)$$

Now

$$M \tilde{T} M^T = M(1 - P)T(1 - P)M^T = \begin{pmatrix} & & & 0 \\ & \hat{T} & & 0 \\ & & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

where (for $i, j \leq N - 1$)

$$\hat{T}_{ij} = T_{ij} - T_{iN} - T_{Nj} + T_{NN} = \tilde{T}_{ij} - \tilde{T}_{iN} - \tilde{T}_{Nj} + \tilde{T}_{NN} = \tilde{T}_{ij} + \sum_{k=1}^{N-1} (\tilde{T}_{ik} + \tilde{T}_{kj}) + \sum_{k,l=1}^{N-1} \tilde{T}_{kl}.$$

Thus

$$\mathbf{n}^T T \mathbf{n} = \frac{\tau}{N} k^2 + (\hat{\mathbf{n}} - \frac{k}{N} \mathbf{e}_{N-1})^T \hat{T} (\hat{\mathbf{n}} - \frac{k}{N} \mathbf{e}_{N-1}).$$

In terms of these quantities we see

$$\begin{aligned} \Theta(\mathbf{z}|T) &= \sum_{\mathbf{n} \in \mathbb{Z}^N} e^{2\pi i \mathbf{n} \cdot \mathbf{z} + i\pi \mathbf{n}^T T \mathbf{n}} = \\ &= \sum_{\mathbf{n} \in \mathbb{Z}^N; \sum_{i=1}^N n_i = k} e^{2\pi i \frac{zk}{N} + i\pi \frac{\tau}{N} k^2 + 2\pi i \sum_{j=1}^{N-1} (n_j - \frac{k}{N}) \hat{z}_j + i\pi \sum_{l,m=1}^{N-1} (n_l - \frac{k}{N}) \hat{T}_{lm} (n_m - \frac{k}{N})} = \\ &= \sum_{k \in \mathbb{Z}} e^{2\pi i \frac{k}{N} z + i\pi \frac{k^2}{N} \tau} \sum_{\hat{\mathbf{n}} \in \mathbb{Z}^{N-1}} e^{2\pi i (\hat{\mathbf{n}} - \frac{k}{N} \mathbf{e}_{N-1}) \cdot \hat{\mathbf{z}} + i\pi (\hat{\mathbf{n}} - \frac{k}{N} \mathbf{e}_{N-1})^T \hat{T} (\hat{\mathbf{n}} - \frac{k}{N} \mathbf{e}_{N-1})} \end{aligned} \quad (81)$$

By writing $k = Nm + l$ with $m \in \mathbb{Z}$ and $l \in \mathbb{Z}_N = \mathbb{Z} \bmod N$ (i.e. $i = 0, 1, \dots, N-1$), one has

$$\Theta(\mathbf{z}|T) = \sum_{l \in \mathbb{Z}_N} \sum_{m \in \mathbb{Z}} e^{2\pi i(m + \frac{l}{N})z + i\pi N\tau(m + \frac{l}{N})^2} \sum_{\hat{\mathbf{n}} \in \mathbb{Z}^{N-1}} e^{2\pi i(\hat{\mathbf{n}} - \frac{l}{N}\mathbf{e}_{N-1})\hat{\mathbf{z}} + i\pi(\hat{\mathbf{n}} - \frac{l}{N}\mathbf{e}_{N-1})^T \hat{T}(\hat{\mathbf{n}} - \frac{l}{N}\mathbf{e}_{N-1})}$$

That is

$$\Theta(\mathbf{z}|T) = \sum_{l \in \mathbb{Z}_N} \theta_{\frac{l}{N}}(z|N\tau) \Theta_l(\hat{\mathbf{z}}|\hat{T}) \quad (82)$$

where $\theta_{\frac{l}{N}}$ is the genus $g = 1$ theta-function introduced earlier while

$$\Theta_l(\hat{\mathbf{z}}|\hat{T}) = \sum_{\hat{\mathbf{n}} \in \mathbb{Z}^{N-1}} e^{2\pi i(\hat{\mathbf{n}} - \frac{l}{N}\mathbf{e}_{N-1})\hat{\mathbf{z}} + i\pi(\hat{\mathbf{n}} - \frac{l}{N}\mathbf{e}_{N-1})^T \hat{T}(\hat{\mathbf{n}} - \frac{l}{N}\mathbf{e}_{N-1})} \equiv \Theta \left[\begin{array}{c} \frac{2l}{N}\mathbf{e}_{N-1} \\ 0 \end{array} \right] (-\hat{\mathbf{z}}|\hat{T}) \quad (83)$$

is defined on a $(N-1)$ -dimensional complex torus. (For $N-1 > 4$ this torus corresponding to \hat{T} , is not necessarily a Jacobian.) Our expression (74) is equivalent to that obtained in [23, 26] upon observing

$$\begin{aligned} \Theta_k(\hat{\mathbf{z}}|\hat{T}) &= \sum_{\mathbf{n} \in \mathbb{Z}^N; \sum_{j=1}^N n_j = k} e^{2\pi i \sum_{j=1}^N n_j \hat{z}_j + i\pi \sum_{j,j'=1}^N n_j \hat{T}_{jj'} n_{j'}} \\ &= \sum_{\mathbf{m} \in (\mathbb{Z} - \frac{k}{N})^{N-1}} e^{2\pi i \sum_{j=1}^{N-1} m_j \hat{z}_j + i\pi \sum_{j,j'=1}^{N-1} m_j \hat{T}_{jj'} m_{j'}} \equiv \Theta_k(\hat{\mathbf{z}}|\hat{T}). \end{aligned} \quad (84)$$

We have therefore established (74) for those models with period matrices satisfying (22). The coefficients Θ_k will appear in our later discussions.

For future reference we note that under the transformation $\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R})$, where $A^t D - C^t B = 1_g$, $A^t C = C^t A$ and $B^t D = D^t B$, we have the arguments of the theta function transforming as

$$\begin{aligned} T &\rightarrow T^\Gamma = (AT + B)(CT + D)^{-1}, \\ z &\rightarrow z^\Gamma = [(CT + D)^{-1}]^t z. \end{aligned} \quad (85)$$

Further, the characteristics transform as

$$\begin{aligned} \epsilon &\rightarrow \epsilon^\Gamma = D\epsilon - C\epsilon' + \frac{1}{2}\text{diag}(CD^t), \\ \epsilon' &\rightarrow \epsilon'^\Gamma = -B\epsilon + A\epsilon' + \frac{1}{2}\text{diag}(AB^t). \end{aligned} \quad (86)$$

For $\Gamma = \begin{pmatrix} M & 0 \\ 0 & M^{-1T} \end{pmatrix}$ we see

$$T^\Gamma = MTM^T = \begin{pmatrix} \hat{T} & 0 \\ 0 & 0 \dots N\tau \end{pmatrix}, \quad z^\Gamma = \begin{pmatrix} \hat{\mathbf{z}} \\ z \end{pmatrix},$$

and our previous discussion has simply used this transformation to put the theta function into a canonical form. Further the particular case

$$\Theta(\mathbf{z}|T) = \zeta (\det T)^{-1/2} e^{-i\pi \mathbf{z}^T \mathbf{T}^{-1} \mathbf{z}} \Theta(T^{-1}\mathbf{z}| -T^{-1}). \quad (87)$$

(for some $\zeta^8 = 1$) leads to

$$\begin{aligned} \Theta(\mathbf{z}|T) &= \sum_{k=0}^{N-1} \Theta \left[\begin{array}{c} \frac{k}{N} \\ 0 \end{array} \right] (z|N\tau) \Theta \left[\begin{array}{c} \frac{k}{N}\mathbf{e}_{N-1} \\ 0 \end{array} \right] (-\hat{\mathbf{z}}|\hat{T}) \\ &= \zeta (\det \hat{T})^{-1/2} (N\tau)^{-1/2} e^{-i\pi \hat{\mathbf{z}} \hat{T}^{-1} \hat{\mathbf{z}} - i\pi \frac{k^2}{N\tau}} \sum_{k=0}^{N-1} \Theta(\frac{z}{N\tau} + \frac{k}{N}|\frac{-1}{N\tau}) \Theta(\hat{T}^{-1}\hat{\mathbf{z}} - \frac{k}{N}\mathbf{e}_{N-1}| -\hat{T}^{-1}). \end{aligned} \quad (88)$$

2.4 Elliptic solutions

We conclude this section by examining some explicit solutions to the integrable systems described earlier. The simplest non degenerate periodic solutions arise in integrable systems with only two interacting particles. Because the center of mass decouples in the Calogero-Moser families we have been considering there is effectively one degree of freedom in this case. We will consider the “periodic” (sine-Gordon) Toda chain and Calogero-Moser models.

For the example of the “periodic” Toda chain with two particles an explicit solution is a simple consequence of the addition formula for the (Weierstrass) elliptic functions

$$\wp(\mu + \xi) + \wp(\mu - \xi) - 2\wp(\mu) = -\frac{\partial^2}{\partial \mu^2} \log(\wp(\mu) - \wp(\xi)). \quad (89)$$

Indeed, consider

$$e^q = A \frac{\sigma(\mu + \omega)}{\sigma(\mu)\sigma(\omega)} e^{-\eta\mu}, \quad (90)$$

where $\mu = Ut$, ω is any of the (half-) periods ($\wp(\mu + 2\omega) = \wp(\mu)$ and $\eta = \zeta(\omega)$) and the constants A and U have yet to be determined. Then

$$\begin{aligned} e^{2q} &= A^2 \frac{\sigma^2(\mu + \omega)}{\sigma^2(\mu)\sigma^2(\omega)} e^{-2\eta\mu} = -A^2 \frac{\sigma(\mu + \omega)\sigma(\mu - \omega)}{\sigma^2(\mu)\sigma^2(\omega)} = A^2 (\wp(\mu) - \wp(\omega)), \\ e^{-2q} &= A^{-2} \frac{1}{\wp(\mu) - \wp(\omega)} = \frac{1}{A^2 H^2} (\wp(\mu + \omega) - \wp(\omega)). \end{aligned} \quad (91)$$

Here $H^2 = (e - e_+)(e - e_-)$, with $e = \wp(\omega)$ and $e_{\pm} = \wp(\omega_{\pm})$, where ω_{\pm} are the remaining half-periods: $\omega + \omega_+ + \omega_- = 0$ modulo the lattice. If one puts $A^4 H^2 = 1$ then

$$\begin{aligned} e^{2q} - e^{-2q} &= \frac{1}{H} (\wp(\mu + \omega) - \wp(\mu)) = \frac{1}{2H} (\wp(\mu + \omega) + \wp(\mu - \omega) - 2\wp(\mu)) = -\frac{1}{2H} \frac{\partial^2}{\partial \mu^2} \log(\wp(\mu) - \wp(\omega)) \\ &= -\frac{1}{2H} \frac{\partial^2}{\partial \mu^2} \log \frac{\sigma(\mu + \omega)\sigma(\omega - \mu)}{\sigma^2(\mu)\sigma^2(\omega)} = -\frac{1}{2H} \frac{\partial^2}{\partial \mu^2} \log e^{2q} = -\frac{1}{H} \frac{\partial^2 q}{\partial \mu^2} = -\frac{1}{HU^2} \frac{d^2 q}{dt^2}. \end{aligned} \quad (92)$$

After introducing Λ via

$$U = \frac{\Lambda}{i\sqrt{H}} \quad (93)$$

formula (92) acquires the “canonical” form

$$\frac{d^2 q}{dt^2} = \Lambda^2 (e^{2q} - e^{-2q}) \quad (94)$$

with Λ^2 playing the role of a coupling constant. We may rescale $\Lambda \rightarrow 1$, in which case these are the equations of motion for

$$h = p^2 + e^{2q} + e^{-2q} \equiv p^2 + 2 \cosh 2q \quad (95)$$

which is the periodic Toda Hamiltonian (33) in the centre of mass frame. (Here $q = q_1 = -q_2$.) The Lax operator for this example is

$$\mathcal{L}(w) = \begin{pmatrix} p & e^{-q} + \frac{1}{w} e^q \\ e^{-q} + we^q & -p \end{pmatrix} \quad (96)$$

and the corresponding spectral curve in this case

$$w + \frac{1}{w} = \lambda^2 - h. \quad (97)$$

The scale parameter Λ may be restored in the above by setting $p \rightarrow p/\Lambda$, $\lambda \rightarrow \lambda/\Lambda$ and $w \rightarrow w/\Lambda^2$. Finally the perturbative limit leading to (6) is obtained by setting $p \rightarrow p/\Lambda$, $\lambda \rightarrow \lambda/\Lambda$, $w \rightarrow w/\Lambda^2$ and also $q \rightarrow q - \log \Lambda$ and taking the $\Lambda \rightarrow 0$ limit. Another convenient representation for the spectral curve (97) is the elliptic parameterization

$$w^3 + hw^2 + w = \lambda^2 w^2, \quad (98)$$

where

$$\begin{aligned} w &= x - e = \wp(z) - e = \wp(z) - \wp(\omega) = -\frac{\sigma(z + \omega)\sigma(z - \omega)}{\sigma^2(z)\sigma^2(\omega)} \\ &= \left(\frac{\sigma(z + \omega)}{\sigma(z)\sigma(\omega)} e^{-\zeta(\omega)z} \right)^2 = \left(\frac{\sigma(z - \omega)}{\sigma(z)\sigma(\omega)} e^{\zeta(\omega)z} \right)^2 \end{aligned} \quad (99)$$

and

$$\lambda = \frac{1}{2} \frac{\wp'(z)}{\wp(z) - e}. \quad (100)$$

In this form $h = e$, $e_+e_- = 1$ and $g_2 = 4(h^2 - 1)$, $g_3 = 4h$. Also $dz = 2\frac{dw}{\lambda w}$. Observe that $\lambda \rightarrow \infty$ as $w \rightarrow 0, \infty$ (equivalently $z \rightarrow \omega, 0$ or $x \rightarrow e, \infty$). Let us denote P_+ to be the point $\lambda \rightarrow \infty$, $w \rightarrow \infty$ and P_- to be the point $\lambda \rightarrow \infty$, $w \rightarrow 0$. Further let $P(1)$ be the point $\lambda = p$, $w = -e^{-2q}$ and $P(0)$ the point $\lambda = -p$, $w = -e^{2q}$.

The solution we have just obtained may be derived from the Baker-Akhiezer function for this problem. The auxiliary linear problem $\mathcal{L}\Psi = \lambda\Psi$ has solution

$$\Psi = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{we^q + e^{-q}}{\lambda + p} \end{pmatrix} \quad (101)$$

while the “conjugate” equation $\hat{\Psi}\mathcal{L} = \lambda\hat{\Psi}$ has solution

$$\hat{\Psi} = (\hat{\psi}_0, \hat{\psi}_1) = \left(\frac{we^q + e^{-q}}{\lambda - p}, 1 \right). \quad (102)$$

Consider the expression

$$\frac{\psi_1}{\psi_0} = \frac{we^q + e^{-q}}{\lambda + p}.$$

From our earlier remarks this vanishes at P_- and because $w \sim \lambda^2$ as $\lambda \rightarrow \infty$ it has a pole at P_+ . Also there is a further zero at $P(1)$ and a pole at $P(0)$, and so we have the divisor of ψ_k

$$(\psi_k) = P(k) - P(0) - kP_+ + kP_-. \quad (103)$$

Now a generic Baker-Akhiezer function should have the same number of poles and zeros. Introducing

$$\psi_k = w^{k/2} \tilde{\psi}_k \quad (104)$$

gives

$$\frac{\psi_1}{\psi_0} = w^{1/2} \frac{\tilde{\psi}_1}{\tilde{\psi}_0} \quad (105)$$

and

$$(\tilde{\psi}_i) = P(i) - P(0). \quad (106)$$

In particular we may express the the Baker-Akhiezer function in terms of the elliptic parameterisation (99) as

$$\tilde{\psi}_i = \text{const.} \frac{\theta(z - z(P(i)))}{\theta(z - z(P(0)))}. \quad (107)$$

We conclude this section with the simplest example of the elliptic Calogero-Moser model, that corresponding to two particles. In the centre of mass frame $h_1^{CM} = p_1 + p_2 = 0$ eqn. (42) turns into

$$\lambda^2 + h - x = 0, \quad (108)$$

where $h = -h_2^{CM} = -p_1p_2 + \wp(q_{12})$, $x = \wp(z)$ and we have removed the m^2 dependence in (42) using the scaling properties (see Appendix A) of the \wp function. This equation says that to any value of x one associates two points of Σ^{CM}

$$\lambda = \pm \sqrt{h - x}, \quad (109)$$

i.e. it describes Σ^{CM} as a double covering of an elliptic curve ramified at the points $x = h$ and $x = \infty$. In fact, $x = h$ corresponds to a *pair* of points distinguished by the sign of y , but $x = \infty$ is one of the branch points, and so the *two* cuts between $x = h$ and $x = \infty$ on each sheet become effectively a single cut between $(h, +)$ and $(h, -)$. The spectral curve Σ^{CM} may therefore be considered as two tori glued along one cut, and so $\Sigma_{N=2}^{CM}$ has genus 2. This is a hyperelliptic curve (for $N = 2$ only!). The two holomorphic 1-differentials on Σ^{CM} ($g = N = 2$) can be chosen to be

$$dv_+ = dz = \frac{dx}{2y} = \frac{\lambda d\lambda}{y} \quad dv_- = \frac{dz}{\lambda} = \frac{dx}{2y\lambda} = \frac{d\lambda}{y} \quad (110)$$

so that

$$dS = 2\lambda dz = \lambda \frac{dx}{y} = \frac{dx}{y} \sqrt{h-x}, \quad (111)$$

and

$$-\frac{\partial dS}{\partial h} \cong \frac{dx}{2y\lambda} = dv_-. \quad (112)$$

The fact that only one of the two holomorphic 1-differentials (110) appears on the right hand side of (112) is related to their different parities with respect to the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ symmetry of Σ^{CM} : $y \rightarrow -y$, $\lambda \rightarrow -\lambda$ and $dv_{\pm} \rightarrow \pm dv_{\pm}$. Since dS has a positive parity, its integrals along two of the four elementary non-contractable cycles on Σ^{CM} automatically vanish, leaving only two non-vanishing quantities a and a^D , as necessary for the SW interpretation. Moreover, the two remaining nonzero periods can be defined in terms of a “reduced” curve of genus $g = 1$

$$Y^2 = (y\lambda)^2 = 4(h-x) \prod_{a=1}^3 (x-e_a), \quad (113)$$

equipped with $dS = (h-x) \frac{dx}{Y}$. This curve arises when directly integrating the equations of motion since after decoupling the free motion of the center of mass we again have a dynamical system with only one degree of freedom. From the conserved energy $h = p^2 + \wp(q)$ we obtain

$$t = \int \frac{dq}{\sqrt{h - \wp(q)}} = \int \frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)(h-x)}} \quad (114)$$

which is exactly the Abel map of the reduced curve (113).

3 The weak coupling limit

In this section we are going to study the weak coupling limit in some detail. First we consider the $\Lambda \rightarrow 0$ limit of the periodic Toda chain, demonstrating that this leads exactly to the open Toda chain or Toda molecule. Here we explicitly derive formulae for the theta-functions (tau-functions) in this limit and show that they appear to be a finite-dimensional analogue of those appearing in matrix models. We discuss their relation with duality in integrable systems and the commutativity of theta-functions [26]. We then turn briefly to the corresponding properties of the weak coupling limit of Calogero-Moser models.

3.1 From the periodic Toda chain to the Toda molecule

The perturbative limit corresponds to $\tau \rightarrow +i\infty$ and $\Lambda \rightarrow 0$. The effect of this will be to simplify the theta function solutions describing the general finite gap situation.

First, in the limit $\tau \rightarrow +i\infty$, the Jacobi theta-functions in (74) turn into exponentials and one finds that

$$\Theta(\mathbf{z}|T) = \sum_{k \in \mathbb{Z}_N} e^{2\pi i \frac{k}{N} z} \Theta_k. \quad (115)$$

We may already deduce an interesting consequence: the ratios of the coefficients $\Theta_k \equiv \Theta_k(\hat{\mathbf{z}}|\hat{T}) \equiv \Theta_k(\tilde{\mathbf{z}}|\tilde{T})$ Poisson commute,

$$\left\{ \frac{\Theta_i}{\Theta_j}, \frac{\Theta_{i'}}{\Theta_{j'}} \right\} = 0, \quad \forall i, j, i', j'. \quad (116)$$

The Poisson bracket here is that corresponding to the symplectic form

$$\Omega^{Toda} = \sum_{i=1}^{N-1} d\hat{z}_i \wedge da_i = \sum_{i=1}^{N-1} d\tilde{q}_i \wedge d\tilde{p}_i, \quad (117)$$

where \tilde{q}_i and \tilde{p}_i are co-ordinates and momenta of the Toda chain and we are working in the centre of mass frame $\tilde{q}_i = q_i - q_{CM}$. The Poisson commutativity of the ratios (116) follows from the solution of the periodic Toda chain [27, 28]

$$e^{\tilde{q}_i} = \frac{\Theta_i}{\Theta_{i-1}}, \quad \frac{\Theta_i}{\Theta_j} = \prod_{k=j+1}^i e^{\tilde{q}_k}. \quad (118)$$

(Here $\Theta_0 = \Theta_N$ and the Θ_k can be thought of as the Toda chain tau-functions depending on the discrete time k , the number of the particle. Observe $\sum_i \tilde{q}_i = 0$ here.) Now because the coordinates \tilde{q}_i obviously Poisson commute, $\{\tilde{q}_i, \tilde{q}_j\} = 0$, we deduce (116). This expression gives a precise formulation of the old expectation that the Toda chain tau-functions Poisson commute with each other. We will return to this point shortly when we discuss duality. Henceforth we shall drop the tilde from the coordinates and momenta for simplicity.

The second set of simplifications arise because the spectral curve (38) degenerates into a rational curve (1)

$$w = P_N(\lambda) = \prod_{i=1}^N (\lambda - a_i) \quad (119)$$

($\sum a_i = 0$) in the perturbative limit $\Lambda \rightarrow 0$. Then upon making the natural choice (for the cylinder) $z = \log w$

$$a_i = \oint_{A_i} \lambda \frac{dw}{w} \quad (120)$$

and we have the basis of (normalised) holomorphic differentials

$$2\pi i d\omega_i = \frac{d\lambda}{\lambda - a_i}. \quad (121)$$

The perturbative period matrix $\tilde{T}_{ij} = \frac{\partial a_i^D}{\partial a_j}$ (where $a_i^D = \oint_{B_i} \lambda \frac{dw}{w}$) is given by

$$-i\pi \tilde{T}_{ij} \equiv -i\pi \tilde{T}_{ij}^{\text{pert}} = \delta_{ij} \sum_{l \neq i} \log \frac{|a_{il}|}{\Lambda} - (1 - \delta_{ij}) \log \frac{|a_{ij}|}{\Lambda}. \quad (122)$$

We shall substitute this into (115) and take the $\Lambda \rightarrow 0$ limit, but first we should be more careful with the various appearances of Λ . We have already mentioned the scalings $h_k \rightarrow h_k/\mu^k$, $w \rightarrow w/\mu^N$ along with $\lambda \rightarrow \lambda/\mu$. Comparison of (115) and the relation between $w \sim e^{z+i\pi\tau}$ and z shows the first two of these scalings to be achieved by $2\pi iz \rightarrow 2\pi iz - N \log \mu$. Now the double scaling limit (where $\Lambda = m e^{i\pi\tau/N}$) is achieved by shifting $2\pi iz \rightarrow 2\pi iz - N^2 \log \Lambda$. (We shall justify this a little later.) Upon noting the τ dependence of the genus one theta function we see $\Theta_k \rightarrow \Lambda^{k^2} \Theta_k$ and overall we must construct

$$\begin{aligned} \bar{\Theta}_k &\equiv \lim_{\Lambda \rightarrow 0} \Lambda^{k^2 - Nk} \Theta_k = \\ &= \sum_{\sum_{i=1}^N n_i = k; \sum_{i < j} (n_i - n_j)^2 = k(N-k)} e^{2\pi i \sum_{j=1}^N n_j \tilde{z}_j + i\pi \sum_{i,j=1}^N n_i n_j ((1 - \delta_{ij}) \log |a_{ij}| - \delta_{ij} \sum_{l \neq i} \log |a_{il}|)} = \\ &= \sum_{\sum_{i=1}^N n_i = k; \sum_{i < j} (n_i - n_j)^2 = k(N-k)} e^{2\pi i \sum_{j=1}^N n_j \tilde{z}_j - i\pi \sum_{i < j=1}^N (n_i - n_j)^2 \log |a_{ij}|}. \end{aligned} \quad (123)$$

The quadratic constraint here appears when the Λ dependence of \tilde{T}_{ij} is taken into account. Now the conditions $\sum n_i = k$ and $\sum_{i < j} (n_i - n_j)^2 = k(N - k)$ can only be satisfied for each $n_i \in \{0, 1\}$, with k of these nonzero. Thus we may write $\bar{\Theta}_k$ as

$$\bar{\Theta}_k = \sum_{|I|=k} \prod_{i \in I} e^{2\pi i \tilde{z}_i} \prod_{j \in \bar{I}} \frac{1}{|a_{ij}|} \quad (124)$$

for some set of indices $I = \{i_1, \dots, i_k\}$, and \bar{I} is the set of indices complementary to I ($n_i = 1$ if $i \in I$ and $n_i = 0$ otherwise). These are expressions for the tau-functions of the *open* Toda chain or the *Toda molecule*.

Let us consider the $N = 2$ version of these formulae. For this case of one degree of freedom we have the hamiltonian (6) $H = p^2 + e^{2q}$. Taking for this example the period matrix

$$-i\pi \|\tilde{T}_{ij}\| = \begin{pmatrix} \log \frac{|a_1 - a_2|}{\Lambda} & -\log \frac{|a_1 - a_2|}{\Lambda} \\ -\log \frac{|a_1 - a_2|}{\Lambda} & \log \frac{|a_1 - a_2|}{\Lambda} \end{pmatrix} \equiv \begin{pmatrix} \log \frac{a}{\Lambda} & -\log \frac{a}{\Lambda} \\ -\log \frac{a}{\Lambda} & \log \frac{a}{\Lambda} \end{pmatrix} \quad (125)$$

formula (123) gives in this case $\bar{\Theta}_0 = 1$ and

$$\bar{\Theta}_1 = \sum_{i+j=1; (i-j)^2=1} e^{(i-j)z - i\pi(i-j)^2 \log a}, \quad (126)$$

where we have substituted $2\pi i \tilde{z}_1 \rightarrow z$, $2\pi i \tilde{z}_2 \rightarrow -z$. Thus

$$\bar{\Theta}_1 = \sum_{i+j=1; (i-j)^2=1} e^{-(i-j)^2 \log a + (i-j)z} = \frac{1}{a}(e^z + e^{-z}). \quad (127)$$

Now $e^{2q} = e^{q_2 - q_1} = (\bar{\Theta}_2 / \bar{\Theta}_1)^2 = 1 / \bar{\Theta}_1^2$ and so we again arrive at the explicit solution to the equation of motion⁷ for the hamiltonian $h = p^2 + e^{2q}$

$$e^q = \frac{\sqrt{h}}{\cosh z}. \quad (128)$$

In the $N = 3$ case eq. (124) gives

$$\bar{\Theta}_1 = e^{2\pi i \tilde{z}_1} \frac{1}{|a_{12}a_{13}|} + e^{2\pi i \tilde{z}_2} \frac{1}{|a_{12}a_{23}|} + e^{2\pi i \tilde{z}_3} \frac{1}{|a_{13}a_{23}|} \quad (129)$$

and one further nontrivial $\bar{\Theta}$ function

$$\begin{aligned} \bar{\Theta}_2 &= e^{2\pi i(\tilde{z}_1 + \tilde{z}_2)} \frac{1}{|a_{13}a_{23}|} + e^{2\pi i(\tilde{z}_1 + \tilde{z}_3)} \frac{1}{|a_{12}a_{23}|} + e^{2\pi i(\tilde{z}_2 + \tilde{z}_3)} \frac{1}{|a_{12}a_{13}|} = \\ &= e^{-2\pi i \tilde{z}_1} \frac{1}{|a_{12}a_{13}|} + e^{-2\pi i \tilde{z}_2} \frac{1}{|a_{12}a_{23}|} + e^{-2\pi i \tilde{z}_3} \frac{1}{|a_{13}a_{23}|}. \end{aligned} \quad (130)$$

There is a convenient determinantal form for formula (124). In general one has that

$$\bar{\Theta}_k = \det_{k \times k} K_{n+m} \Big|_{n,m=1,\dots,k} \quad (131)$$

where the “moment matrix” $K_{nm} = K_{n+m}$ is defined as an average with respect to $\bar{\Theta}_1$, i.e.

$$K_n = \langle a^n \rangle_1 = \sum_{i=1}^N e^{Z_i} a_i^{n-2}, \quad e^{Z_i} \equiv e^{2\pi i \tilde{z}_i} \prod_{j \neq i} \frac{1}{|a_{ij}|}, \quad \bar{\Theta}_1 = \sum_{i=1}^N e^{Z_i}. \quad (132)$$

The proof is similar to that in matrix models [29]. Indeed,

$$\begin{aligned} \bar{\Theta}_k &= \sum_{|I|=k} \prod_{i \in I} e^{2\pi i \tilde{z}_i} \prod_{j \in \bar{I}} \frac{1}{|a_{ij}|} = \sum_{|I|=k} \prod_{i \in I} e^{2\pi i \tilde{z}_i} \left(\prod_{j \in \bar{I}} \frac{1}{|a_{ij}|} \prod_{j \in I \setminus \{i\}} \frac{1}{|a_{ij}|} \prod_{j \in I \setminus \{i\}} |a_{ij}| \right) \\ &= \sum_{|I|=k} \prod_{i \in I} \left(e^{Z_i} \prod_{j \in I \setminus \{i\}} |a_{ij}| \right) = \sum_{|I|=k} \prod_{i \in I} e^{Z_i} \prod_{i \neq j \in I} |a_{ij}| = \sum_{|I|=k} \prod_{i \in I} e^{Z_i} \prod_{i < j; i,j \in I} a_{ij}^2 \end{aligned} \quad (133)$$

looks like a “discrete” analogue of the tau-function of the “forced” Toda chain hierarchy which plays a central role in matrix models [29]. Now⁸ for *any* coefficients C_i

$$\begin{aligned} \sum_{|I|=k} \prod_{i \in I} C_i \prod_{i < j; i,j \in I} a_{ij}^2 &\equiv \sum_{I: i_1 < \dots < i_k} C_{i_1} \dots C_{i_k} \prod_{i_n < i_m} (a_{i_n} - a_{i_m})^2 \\ &= \sum_{i_1 < \dots < i_k} C_{i_1} \dots C_{i_k} \det_{k \times k} a_i^{n-1} \Big|_{i \in I; n=1,\dots,k} \det_{k \times k} a_i^{m-1} \Big|_{i \in I; m=1,\dots,k} \\ &= \det_{k \times k} \left(\sum_{i=1}^N C_i a_i^{n+m-2} \right) \Big|_{n,m=1,\dots,k}. \end{aligned} \quad (135)$$

Substituting $C_i = e^{Z_i}$, one arrives at eqs. (131), (132)

$$\bar{\Theta}_k = \sum_{I: i_1 < \dots < i_k} e^{Z_{i_1} + \dots + Z_{i_k}} \prod_{i_n < i_m} (a_{i_n} - a_{i_m})^2 = \det_{k \times k} \left(\sum_{i=1}^N e^{Z_i} a_i^{n+m-2} \right) \Big|_{n,m=1,\dots,k} = \det_{k \times k} K_{n+m}. \quad (136)$$

These formulae in fact coincide with the solution of the N -particle *open* Toda chain problem in terms of the action-angle variables discussed in [17].

⁷It is interesting to point out that the equation of motion for the co-ordinate $Q = a = \sqrt{h}$ in the dual system $H = \frac{\cosh P}{Q}$ coincides with the equation of motion for the rational Calogero model, $\ddot{Q} = \frac{1}{Q^3}$. Similar dualities are known for the Calogero models and their “relativistic” counterparts.

⁸This is a particular case of the Cauchy-Binet formula for the $k \times N$ ($k \leq N$) rectangular matrices A_{ni} and B_{im} ($i = 1, \dots, N$, $n, m = 1, \dots, k$)

$$\det_{k \times k} \left(\sum_{i=1}^N A_{ni} B_{im} \right) \Big|_{n,m=1,\dots,k} = \sum_{i_1 < \dots < i_k} \det_{k \times k} A_{ni} \Big|_{i \in I; n=1,\dots,k} \det_{k \times k} B_{im} \Big|_{i \in I; m=1,\dots,k}. \quad (134)$$

Let us relate these formulae to the Baker-Akhiezer function. First, we have defined the angle variables as co-ordinates of the Jacobian of genus $g = N$ curve Σ^N by (19). For the holomorphic differentials (121) we have

$$2\pi i z_i = \sum_{k=1}^N \int_{P_0}^{P_k} \frac{d\lambda}{\lambda - a_i} = \sum_{n=0}^{\infty} a_i^n \sum_{k=1}^N \int_{P_0}^{P_k} \frac{d\lambda}{\lambda^{n+1}} \Big|_{\lambda_k \rightarrow \tilde{\lambda}(P_0)=\infty} N \log \lambda + \sum_{n=1}^{\infty} a_i^n T_n. \quad (137)$$

Here $\lambda_k \equiv \lambda(P_k)$ and

$$T_n = -\frac{1}{n} \sum_{k=1}^N \lambda_k^{-n} \quad (138)$$

is the so called Miwa parameterization of the “canonical” Toda times. The logarithmically divergent first term here is absorbed into the renormalisation $z_i \rightarrow z_i - \frac{N}{2\pi i} \log \Lambda$. Summing over i yields $z \rightarrow z - \frac{N^2}{2\pi i} \log \Lambda$ and the double scaling limit we have already mentioned. The remaining “finite” part of the right hand side here describes the Toda chain dynamics with respect to the various higher times t_n , $n \geq 1$

$$2\pi i z_i = a_i t + \sum_{n>1} a_i^n t_n + z_i^{(0)}. \quad (139)$$

The ordinary time $t_1 = t$ here corresponds to the evolution with respect to the Hamiltonian quadratic in the momenta (or quadratic in canonical action variables)

$$2\pi i z_i = a_i t + z_i^{(0)}. \quad (140)$$

The Baker-Akhiezer functions on the degenerate spectral curve (119) may now be defined by the following analytic requirements: It is the set of functions $\psi_k = \psi_k(\lambda)$, $k = 0, \dots, N-1$ which have exactly k zeroes on the rational curve (119) and a single pole of order k at $\lambda = \infty$. For $k \geq N$ they can be defined by $\psi_{k+N} = w\psi_k$. This means each $\psi_k(\lambda)$, $k = 0, \dots, N-1$ may be represented by a polynomial with k (finite) zeroes. Thus ψ_k can be constructed as a linear combination of $\prod_{i_1 < \dots < i_k} (\lambda - a_{i_l}) = \prod_{j \in I} (\lambda - a_j)$. In fact one has

$$\psi_k(\lambda) = \lambda^k \frac{\bar{\Theta}_k \left(z_i - \frac{1}{2i\pi} \sum_{n \geq 1} \frac{a_i^n}{n \lambda^n} \right)}{\bar{\Theta}_k(z_i)} = \frac{\sum_{|I|=k} \prod_{i \in I} e^{2\pi i z_i} (\lambda - a_i) \prod_{j \in I} \frac{1}{|a_{ij}|}}{\sum_{|I|=k} \prod_{i \in I} e^{2\pi i z_i} \prod_{j \in I} \frac{1}{|a_{ij}|}}. \quad (141)$$

Thus, for example

$$\begin{aligned} \psi_1(\lambda) &= \lambda \frac{\bar{\Theta}_1 \left(z_i - \frac{1}{2i\pi} \sum_{n \geq 1} \frac{a_i^n}{n \lambda^n} \right)}{\bar{\Theta}_1(z_i)} = \frac{\sum_{i=1}^N (\lambda - a_i) e^{2\pi i z_i} \prod_{j \neq i} \frac{1}{|a_{ij}|}}{\sum_{i=1}^N e^{2\pi i z_i} \prod_{j \neq i} \frac{1}{|a_{ij}|}} = \\ &= \lambda - \frac{\sum_{i=1}^N a_i e^{2\pi i z_i} \prod_{j \neq i} \frac{1}{|a_{ij}|}}{\sum_{i=1}^N e^{2\pi i z_i} \prod_{j \neq i} \frac{1}{|a_{ij}|}} \equiv \lambda - \langle a \rangle \end{aligned} \quad (142)$$

with $\langle a \rangle \equiv \langle a^3 \rangle_1 / \langle a^2 \rangle_1$. We note that these Baker-Akhiezer functions satisfy

$$\sum_{i=1}^N e^{Z_i} \psi_k(a_i) = 0$$

which is a consequence of the identity

$$\sum_{i=1}^N \prod_{j \neq i} \frac{1}{a_i - a_j} = 0.$$

(This may be established using Lagrange interpolation: $1 = \sum_{i=1}^N \prod_{j \neq i} \frac{x - a_j}{a_i - a_j}$.) One may also easily write the equations of motion for the zeroes of the Baker-Akhiezer functions $\psi_k(\gamma_i) = 0$:⁹

$$\frac{\partial \gamma_i}{\partial t} = \frac{\prod_{j=1}^N (\gamma_i - a_j)}{\prod_{j \neq i} (\gamma_i - \gamma_j)}, \quad \frac{\partial \gamma_i}{\partial t_n} = \frac{\gamma_i^{n-1} \prod_{j=1}^N (\gamma_i - a_j)}{\prod_{j \neq i} (\gamma_i - \gamma_j)}. \quad (144)$$

⁹They can be considered, for example, as a degeneration of corresponding equations for the zeroes of the Baker-Akhiezer function of periodic Toda chain

$$\frac{\partial \gamma_i}{\partial t} = \frac{\sqrt{P_N^2(\gamma_i) - 4\Lambda^{2N}}}{\prod_{j \neq i} (\gamma_i - \gamma_j)}, \quad \frac{\partial \gamma_i}{\partial t_n} = \frac{\gamma_i^{n-1} \sqrt{P_N^2(\gamma_i) - 4\Lambda^{2N}}}{\prod_{j \neq i} (\gamma_i - \gamma_j)}. \quad (143)$$

Finally let us note that the conserved Hamiltonians are straightforwardly given in terms of the minors of the Lax matrix. The Lax representation gives

$$\det(\lambda - \mathcal{L}) = \prod_{i=1}^N (\lambda - a_i) = \sum_{k=0}^N \lambda^{N-k} (-1)^k h_k \quad (145)$$

with $h_0 \equiv 1$, and

$$h_k = \sum_{i_1 < \dots < i_k} a_{i_1} \dots a_{i_k} \quad k = 1, \dots, N. \quad (146)$$

On the other hand, since (for *any* matrix \mathcal{L})

$$\begin{aligned} \det(\lambda - \mathcal{L}) &= \\ &= \lambda^N - \lambda^{N-1} \sum_i \mathcal{L}_{ii} + \lambda^{N-2} \sum_{i < j} \det \begin{pmatrix} \mathcal{L}_{ii} & \mathcal{L}_{ij} \\ \mathcal{L}_{ji} & \mathcal{L}_{jj} \end{pmatrix} - \lambda^{N-3} \sum_{i < j < k} \det \begin{pmatrix} \mathcal{L}_{ii} & \mathcal{L}_{ij} & \mathcal{L}_{ik} \\ \mathcal{L}_{ji} & \mathcal{L}_{jj} & \mathcal{L}_{jk} \\ \mathcal{L}_{ki} & \mathcal{L}_{kj} & \mathcal{L}_{kk} \end{pmatrix} + \dots = \\ &= \lambda^N - \lambda^{N-1} \text{Tr} \mathcal{L} + \sum_{k=2}^N (-1)^k \lambda^{N-k} \sum_{i_1 < \dots < i_k} \det \begin{pmatrix} \mathcal{L}_{i_1 i_1} & \mathcal{L}_{i_1 i_2} & \dots & \mathcal{L}_{i_1 i_k} \\ \mathcal{L}_{i_2 i_1} & \mathcal{L}_{i_2 i_2} & \dots & \mathcal{L}_{i_2 i_k} \\ \vdots & & \ddots & \vdots \\ \mathcal{L}_{i_k i_1} & \mathcal{L}_{i_k i_2} & \dots & \mathcal{L}_{i_k i_k} \end{pmatrix} \end{aligned} \quad (147)$$

we have that

$$h_k = \sum_{i_1 < \dots < i_k} a_{i_1} \dots a_{i_k} = \sum_{i_1 < \dots < i_k} \det \begin{pmatrix} \mathcal{L}_{i_1 i_1} & \mathcal{L}_{i_1 i_2} & \dots & \mathcal{L}_{i_1 i_k} \\ \mathcal{L}_{i_2 i_1} & \mathcal{L}_{i_2 i_2} & \dots & \mathcal{L}_{i_2 i_k} \\ \vdots & & \ddots & \vdots \\ \mathcal{L}_{i_k i_1} & \mathcal{L}_{i_k i_2} & \dots & \mathcal{L}_{i_k i_k} \end{pmatrix}. \quad (148)$$

3.2 Theta-functions for the Calogero-Moser models and their Poisson Commutativity

In the preceding discussion of the Toda chain we saw that the ratios¹⁰ of the theta functions Θ_k Poisson commuted by appealing to the explicit form of the equations of motion. In fact a more general result holds for the Calogero-Moser family that we shall now describe. We have seen that the theta functions for the Calogero-Moser family satisfy (74). We wish to show that for appropriate solutions (and for all k, l, m, n)

$$0 = \left\{ \frac{\Theta_k}{\Theta_l}, \frac{\Theta_m}{\Theta_n} \right\} \iff \Theta_l S_{kmn} = \Theta_k S_{lmn} \quad (149)$$

where

$$S_{lmn} \equiv \Theta_l \{\Theta_m, \Theta_n\} + \Theta_m \{\Theta_n, \Theta_l\} + \Theta_n \{\Theta_l, \Theta_m\} = S_{mnl} = -S_{lnm}.$$

Indeed (upon setting $l = n$ here and using the antisymmetry of S) we see that (for all k, m, n) $S_{kmn} = 0$ or

$$0 = S_{kmn} = \sum_{a=1}^N \begin{vmatrix} \frac{\Theta_k}{\partial q_a} & \frac{\Theta_m}{\partial q_a} & \frac{\Theta_n}{\partial q_a} \\ \frac{\partial \Theta_k}{\partial p^a} & \frac{\partial \Theta_m}{\partial p^a} & \frac{\partial \Theta_n}{\partial p^a} \end{vmatrix}.$$

Such addition formulae are very restrictive [30] and closely connected with integrable systems.

We establish for the Calogero-Moser system the commutativity (149) in the following way [26]. According to [19] the equation

$$0 = \Theta(\mathbf{z}|T) = \sum_{i \in \mathbb{Z}_N} \theta_{\frac{i}{N}}(z|N\tau) \Theta_i \quad (150)$$

¹⁰It is perhaps helpful to recall at this point that theta-functions are used to embed a curve into some projective space giving the *inhomogeneous* co-ordinates of the embedding. Their ratios may be considered as “normal” or *homogeneous* co-ordinates.

(as an equation on the z -torus) has exactly N zeroes $\frac{z}{N} = q_1, \dots, q_k$. As a consequence one gets a system of linear equations

$$\sum_{i=1}^N \theta_{\frac{i}{N}}(Nq_j|N\tau) \Theta_i = 0 \quad j = 1, \dots, N. \quad (151)$$

The system should have nontrivial solutions, i.e. $\det_{ij} \theta_{\frac{i}{N}}(Nq_j|N\tau) = 0$, which effectively reduces the number of degrees of freedom from N to $N - 1$. Then (151) can be rewritten as

$$\sum_{\substack{i=1 \\ i \neq i_0}}^{N-1} \theta_{\frac{i}{N}}(Nq_j|N\tau) \frac{\Theta_i}{\Theta_{i_0}} = \theta_{\frac{i_0}{N}}(Nq_j|N\tau), \quad \forall i_0; \quad j = 1, \dots, N-1, \quad (152)$$

and so using Cramers rule

$$\frac{\Theta_i}{\Theta_{i_0}} = \frac{\det_{k \neq i_0, i \rightarrow i_0; j=1, \dots, N-1} \theta_{\frac{k}{N}}(Nq_j|N\tau)}{\det_{k \neq i_0; j=1, \dots, N-1} \theta_{\frac{k}{N}}(Nq_j|N\tau)}. \quad (153)$$

Therefore, the ratios $\frac{\Theta_i}{\Theta_j}$ depend only on the co-ordinate q_k , $k = 1, \dots, N$ of the Calogero-Moser particles and so obviously Poisson commute with respect to the Calogero-Moser symplectic structure

$$\Omega^{CM} = \sum_{i=1}^N dq_i \wedge dp_i = \sum_{i=1}^N dz_i \wedge da_i \quad (154)$$

restricted to $\sum_{j=1}^N q_j = \text{const}$. The latter condition comes from the vanishing of the determinant

$$\det_{ij} \theta_{\frac{i}{N}}(Nq_j|N\tau) = 0.$$

Indeed, using the θ -function identities described earlier coming from Wick's theorem [22, 21]

$$\det_{ij} \theta_{\frac{i}{N}}(Nq_j|N\tau) \sim \theta_{\Sigma \frac{i}{N}} \left(N \sum_k q_k | N\tau \right) \prod_{i < j} \theta_*(Nq_i - Nq_j | N\tau) \quad (155)$$

and we can compute (153) explicitly; the vanishing of the determinant corresponds to $\sum_{j=1}^N q_j$ being a zero of the theta function $\theta_{\Sigma \frac{i}{N}}$ (with characteristic being the sum of the characteristics $\frac{i}{N}$). This centre of mass constraint is also equivalent to (23), $\sum_{j=1}^N a_j = \text{const}$. A consequence of the ratios $\frac{\Theta_i}{\Theta_j}$ depending only on the co-ordinates of the integrable system is that they may be used to construct a set of independent hamiltonians for a dual system [23, 26].

3.3 The Perturbative Limit of the Calogero-Moser Models

The elliptic Calogero-Moser model degenerates in the perturbative limit of the SW theory $\tau \rightarrow +i\infty$ giving rise to the well-known trigonometric Calogero-Moser-Sutherland model. The solution in terms of the action-angle variables is a direct generalisation of the open Toda chain case and may be presented in terms of the dual rational Ruijsenaars-Schneider model. The salient features are as follows.

The Lax operator

$$\mathcal{L}_{ij} = p_i \delta_{ij} + m(1 - \delta_{ij}) \frac{1}{\sinh q_{ij}} \quad (156)$$

may be considered as a limiting case of the Lax operator of the trigonometric Ruijsenaars-Schneider model (50) in the same way as we derived the Lax operator for the elliptic Calogero-Moser model from its Ruijsenaars-Schneider counterpart. The spectral curve for the model is a minor modification of (119). Indeed, from (68), in the limit $\tau \rightarrow \infty$ one gets (for appropriate imaginary period)

$$\frac{\sinh \frac{1}{2} (z - m \frac{\partial}{\partial \lambda \tau})}{\sinh(\frac{1}{2} z)} P_N(\lambda') = 0. \quad (157)$$

Upon introducing $w = e^z$, one may express this as (8)

$$w = \frac{P_N(\lambda)}{P_N(\lambda - m)} = \prod_{i=1}^N \frac{\lambda - a_i}{\lambda - a_i - m}. \quad (158)$$

Again for $su(N)$ we have $\sum a_i = 0$. The spectral curve is equipped with a generating differential $dS = \lambda \frac{dw}{w}$.

The period matrix in this case is given by [24]

$$-i\pi T_{jk}^{Pert.} = -i\pi\tau\delta_{jk} + \frac{1}{2}\delta_{jk} \sum_{r \neq j} \log \frac{a_{rj}^2}{(a_{rj} + m)(a_{rj} - m)} - \frac{1}{2}(1 - \delta_{jk}) \log \frac{a_{jk}^2}{(a_{jk} + m)(a_{jk} - m)} \quad (159)$$

where the first term corresponds to the bare coupling of the elliptic Calogero-Moser model, the modulus of the base torus $\tau = \theta + \frac{i}{g_{YM}^2}$. In the perturbative limit $\tau \rightarrow +i\infty$ (159) this is renormalised and remains finite. In this limit one gets from (82)

$$\Theta(\mathbf{z}|T) = \sum_{k=0}^N e^{\frac{2\pi i k z}{N}} e^{\frac{i\pi k^2 \tau}{N}} \bar{\Theta}_k \quad (160)$$

so that, finally, the “tau-functions” of the trigonometric Calogero models may be introduced as

$$\bar{\Theta}_k = \sum_{|I|=k} \prod_{i \in I} e^{2\pi i z_i} \prod_{j \in \bar{I}} f(a_{ij}), \quad f(a) = \sqrt{1 - \frac{m^2}{a^2}}. \quad (161)$$

4 The Strong coupling limit

4.1 Solitonic solutions of the periodic Toda chain

The quantum moduli space of the 4D pure $su(N)$ $\mathcal{N} = 2$ SYM has N maximally singular points at which $N - 1$ monopoles become simultaneously massless. These are the confining vacua of an $\mathcal{N} = 1$ theory [1, 31, 32]. At these points the dual variables a_i^D are the appropriate variables to describe the prepotential. The $\mathcal{N} = 2$ spectral curve (32) at these points is described in terms of a Chebyshev polynomial $P_N^{\text{Chebyshev}}(\lambda)$ defined by $P_N^{\text{Chebyshev}}(2 \cos v) = 2 \cos(Nv)$. With $w = e^z$ and this choice of polynomial we see (32) turns into

$$P_N^{\text{Chebyshev}}(\lambda) = 2 \cosh z. \quad (162)$$

From the definition of the Chebyshev polynomial it is clear $\lambda = 2 \cosh(\frac{z}{N})$ is a solution of (162) as indeed is $\lambda = 2 \cosh(\frac{z}{N} + i\frac{2\pi k}{N})$ ($k = 0, \dots, N - 1$). These are the $\mathcal{N} = 1$ points of the theory, related by a \mathbf{Z}_N symmetry, and we will focus on the first of these in performing our analysis.

The hyperelliptic form (35) of the spectral curve (recall $y = w - 1/w$) is now

$$y^2 = P_N(\lambda)^2 - 4 = (\lambda^2 - 4)Q(\lambda)^2 \quad (163)$$

where the roots of polynomial $Q(\lambda)$ are given by

$$Q(\lambda) = \prod_{j=1}^{N-1} (\lambda - 2 \cos \frac{\pi j}{N}). \quad (164)$$

This is a “solitonic” curve in the periodic Toda chain: if we express the curve as $y^2 = \prod_{j=1}^{2N+2} (\lambda - e_k)$ we see that $e_{2k} = e_{2k+1} = \cos \pi k/N$ ($k = 1, \dots, N - 1$) and the corresponding \mathbf{B} periods have collapsed; $e_1 = 2 = -e_{2N+2}$ are single branch points.

Let us now introduce a new variable by

$$\lambda = 2 \cosh \frac{z}{N} \equiv \xi + \xi^{-1}. \quad (165)$$

Now (165) maps the 2-sheeted cover of the λ -plane $y = \sqrt{\lambda^2 - 4}$ to a cylinder with co-ordinate ξ . Thus eqs. (162), (163) describe analytically a cylinder with $N - 1$ distinguished pairs of points. With these coordinates our differentials on the curve now take the form

$$\frac{\lambda^{k-1} d\lambda}{y} = \frac{\lambda^{k-1} d\lambda}{\sqrt{\lambda^2 - 4} \prod_{j=1}^{N-1} (\lambda - 2 \cos \frac{\pi j}{N})} = \frac{d\xi \xi^{N-2} \left(\xi + \frac{1}{\xi}\right)^{k-1}}{\prod_{j=1}^{N-1} \left(\xi - e^{\frac{i\pi j}{N}}\right) \left(\xi - e^{-\frac{i\pi j}{N}}\right)}, \quad k = 1, \dots, N - 1. \quad (166)$$

For a non-degenerate curve these were holomorphic but now they acquire simple poles at the points $\xi_j^+ = e^{\frac{i\pi j}{N}}$ and $\xi_j^- = e^{-\frac{i\pi j}{N}}$ (and have singularities at the infinities of the cylinder (165) $\xi = 0, \infty$). The canonical basis in the space of differentials (166) can be chosen as ($j = 1, \dots, N-1$)

$$\begin{aligned} d\omega_j^D &= \frac{\sin \frac{\pi j}{N}}{\pi} \frac{d\xi}{\left(\xi - e^{\frac{i\pi j}{N}}\right) \left(\xi - e^{-\frac{i\pi j}{N}}\right)} \equiv \frac{\sin \frac{\pi j}{N}}{\pi} \frac{d\xi}{(\xi - \xi_j^+) (\xi - \xi_j^-)} = \\ &= \frac{1}{2\pi i} \left(\frac{d\xi}{\xi - \xi_j^+} - \frac{d\xi}{\xi - \xi_j^-} \right) = \frac{1}{2\pi i} d \log \frac{\xi - \xi_j^+}{\xi - \xi_j^-}. \end{aligned} \quad (167)$$

These differentials are normalised to the **B**-cycles, here the cycles around the marked points ξ_j^\pm ,

$$\oint_{B_i} d\omega_j^D = \oint_{\xi_i^+} d\omega_j^D = - \oint_{\xi_i^-} d\omega_j^D = \delta_{ij}, \quad (168)$$

while certain of the **A**-periods ($\oint_{A_j} d\omega_j^D = \int_{\xi_j^-}^{\xi_j^+} d\omega_j^D$) diverge, the others ($j \neq k$) being given by

$$T_{jk}^D = \oint_{A_j} d\omega_k^D = \frac{1}{2\pi i} \log \frac{\sin^2 \frac{\pi}{2N} (j-k)}{\sin^2 \frac{\pi}{2N} (j+k)} = \frac{1}{i\pi} \log \frac{\sin \frac{\pi}{2N} |j-k|}{\sin \frac{\pi}{2N} (j+k)}. \quad (169)$$

Using this expression one may show that T^D (169) satisfies the Edelstein-Mas [10] identity

$$\sum_{k=1}^{N-1} \sin \frac{\pi k i'}{N} \sin \frac{\pi k j'}{N} \sum_{i,j=1}^N \tilde{T}_{ij}^{\text{pert}}(a_l \rightarrow 2 \cos \frac{\pi(l - \frac{1}{2})}{N}) \cos \frac{\pi k(i - \frac{1}{2})}{N} \cos \frac{\pi k(j - \frac{1}{2})}{N} = \frac{N^2}{4} T_{i'j'}^D. \quad (170)$$

This conjecture of Edelstein and Mas is proven in Appendix B. We note that an equivalent expression to (169) was also established in [33] where an interesting investigation of the $\mathcal{N} = 1$ degenerations of the “multisoliton” solutions is presented.

The Abel map in the present setting is

$$z_j^D = \sum_{k=1}^{N-1} \int^{\xi_k} d\omega_j^D = \frac{1}{2\pi i} \sum_{k=1}^{N-1} \log \frac{\xi_k - \xi_j^+}{\xi_k - \xi_j^-} \equiv \frac{1}{\pi} \sum_{n=1}^{\infty} t_n \sin \frac{\pi j n}{N} \quad (171)$$

where $\boldsymbol{\xi} = \{\xi_k\}$, $k = 1, \dots, N-1$ is the set of $N-1$ points which define the Abel map ¹¹. The asymptotics at “infinity” are given by

$$\begin{aligned} z_j^D \Big|_{\boldsymbol{\xi} \rightarrow \infty} &= \frac{1}{2\pi i} \sum_k \int^{\xi_k} d \log \frac{\xi - \xi_j^+}{\xi - \xi_j^-} = \frac{1}{2\pi i} \sum_k \left(\log \left(1 - \frac{\xi_j^+}{\xi_k} \right) - \log \left(1 - \frac{\xi_j^-}{\xi_k} \right) \right) \\ &= -\frac{1}{2\pi i} \sum_{n=1}^{\infty} ((\xi_j^+)^n - (\xi_j^-)^n) \sum_k \frac{1}{n \xi_k^n} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \sin \frac{\pi j n}{N} \sum_k \frac{1}{n \xi_k^n} \end{aligned} \quad (172)$$

and

$$\begin{aligned} z_j^D \Big|_{\boldsymbol{\xi} \rightarrow 0} &= \frac{1}{2\pi i} \sum_k \int^{\xi_k} d \log \frac{\xi - \xi_j^+}{\xi - \xi_j^-} = \frac{1}{2\pi i} \sum_k \int^{\xi_k} d \log \frac{1 - \frac{\xi}{\xi_j^+}}{1 - \frac{\xi}{\xi_j^-}} = \frac{1}{2\pi i} \sum_k \left(\log \left(1 - \frac{\xi_k}{\xi_j^+} \right) - \log \left(1 - \frac{\xi_k}{\xi_j^-} \right) \right) \\ &= \frac{1}{2\pi i} \sum_{n=1}^{\infty} ((\xi_j^+)^n - (\xi_j^-)^n) \sum_k \frac{\xi_k^n}{n} = \frac{1}{\pi} \sum_{n=1}^{\infty} \sin \frac{\pi j n}{N} \sum_k \frac{\xi_k^n}{n} \end{aligned} \quad (173)$$

provided

$$t_n \Big|_{\boldsymbol{\xi} \rightarrow \infty} = - \sum_k \frac{1}{n \xi_k^n}, \quad t_n \Big|_{\boldsymbol{\xi} \rightarrow 0} = \sum_k \frac{\xi_k^n}{n}. \quad (174)$$

¹¹The Abel map here depends on the genus $N-1$ of the smooth curve (32) that arises from the double scaling limit of the genus N elliptic Calogero-Moser curve. It is interesting to point out that in the perturbative limit the basis of differentials (121) of the perturbative curve remembers even more – that the original curve came from the elliptic Calogero-Moser model and had genus N .

We observe that the relation (171) with $t_{n \geq 2} = 0$,

$$z_j^D = t_1 \sin \frac{\pi j}{N} \quad (175)$$

coincides with the vacuum value of the string tension, or monopole condensate, proportional to the SUSY breaking parameter t_1 .

Naively, in the strong coupling limit $\tilde{T}_{ij} \rightarrow 0$ (and $T_{ii}^D \rightarrow \infty$) the Θ -functions (83) turn into

$$\Theta_k \stackrel{\tilde{T}_{ij} \rightarrow 0}{=} \sum_{\sum_{j=1}^N n_j = k} e^{2\pi i \sum n_j \tilde{z}_j} \quad (176)$$

which is not a well-defined object. The resolution to this may be seen by considering the $N = 2$ case. Here one appears to have

$$\begin{aligned} \Theta_0 &= \sum_{n_1+n_2=0} e^{2\pi i(n_1 \tilde{z}_1 + n_2 \tilde{z}_2)} = \sum_n e^{2\pi i n(\tilde{z}_1 - \tilde{z}_2)} \stackrel{\tilde{z}_1 = -\tilde{z}_2 \equiv \frac{Z}{2}}{=} \sum_n e^{2\pi i n Z} = \theta_3(Z|\hat{T} \rightarrow 0) \\ \Theta_1 &= \sum_{n_1+n_2=1} e^{2\pi i(n_1 \tilde{z}_1 + n_2 \tilde{z}_2)} = \sum_{n_1+n_2=1} e^{i\pi(n_1-n_2)Z} \stackrel{n_1-n_2=n \text{ odd}}{=} \sum_{n \text{ odd}} e^{i\pi n Z} = \theta_2(Z|\hat{T} \rightarrow 0) \end{aligned} \quad (177)$$

and, naively

$$e^q = \frac{\Theta_1}{\Theta_0} = \frac{\theta_2(Z|\hat{T} \rightarrow 0)}{\theta_3(Z|\hat{T} \rightarrow 0)} = 2 \cos \pi Z. \quad (178)$$

This appears to have nothing in common with the desired solitonic solution given by integrating the equations of motion

$$\int dt = \int \frac{dq}{\sqrt{h - e^{2q} - e^{-2q}}} \stackrel{h=2}{=} i \int \frac{dq}{e^q - e^{-q}} \quad (179)$$

in the solitonic limit $h \rightarrow 2$, which gives rise to

$$e^q = i \tan(t - t_0) \quad (180)$$

(for an appropriate constant of integration t_0). However the correct answer does appear after both a shift $Z = \bar{Z} - \frac{1}{2}$ and a *modular transform* $\hat{T} \rightarrow -\frac{1}{\hat{T}} \equiv T^D$. Then one has

$$e^q = \frac{\Theta_1}{\Theta_0} = \frac{\theta_1(\bar{Z}|\hat{T} \rightarrow 0)}{\theta_4(\bar{Z}|\hat{T} \rightarrow 0)} = i \frac{\theta_1(\bar{Z}^D|T^D)}{\theta_2(\bar{Z}^D|T^D)} \stackrel{T^D \rightarrow \infty}{=} i \tan \pi \bar{Z}^D + \mathcal{O}(e^{2\pi i T^D}) \quad (181)$$

where $\bar{Z}^D = \frac{\bar{Z}}{\hat{T}} = -T^D \bar{Z} = \frac{Z}{\hat{T}} + \frac{1}{2\hat{T}}$.

The combination of both a shift and a modular transformation appears to work in general. Using the modular properties of theta-functions we have

$$\begin{aligned} \Theta_k(\hat{\mathbf{z}}|\hat{T}) &= \sum_{\mathbf{m} \in \mathbb{Z}_{N-1}} e^{2\pi i(\mathbf{m} - \frac{k}{N}\mathbf{e})\hat{\mathbf{z}} + i\pi(\mathbf{m} - \frac{k}{N}\mathbf{e})\hat{T}(\mathbf{m} - \frac{k}{N}\mathbf{e})} \\ &= e^{-2\pi i \frac{k}{N}\mathbf{e}\hat{\mathbf{z}} + i\pi \frac{k^2}{N^2}\mathbf{e}\hat{T}\mathbf{e}} \sum_{\mathbf{m} \in \mathbb{Z}_{N-1}} e^{2\pi i \mathbf{m}(\hat{\mathbf{z}} - \frac{k}{N}\hat{T}\mathbf{e}) + i\pi \mathbf{m}\hat{T}\mathbf{m}} = e^{-2\pi i \frac{k}{N}\mathbf{e}\hat{\mathbf{z}} + i\pi \frac{k^2}{N^2}\mathbf{e}\hat{T}\mathbf{e}} \Theta\left(\hat{\mathbf{z}} - \frac{k}{N}\hat{T}\mathbf{e} \middle| \hat{T}\right) \\ &= e^{-2\pi i \frac{k}{N}\mathbf{e}\hat{\mathbf{z}} + i\pi \frac{k^2}{N^2}\mathbf{e}\hat{T}\mathbf{e} - i\pi(\hat{\mathbf{z}} - \frac{k}{N}\hat{T}\mathbf{e})\frac{1}{\hat{T}}(\hat{\mathbf{z}} - \frac{k}{N}\hat{T}\mathbf{e})} \left(\det \hat{T}\right)^{-\frac{1}{2}} \Theta\left(\hat{T}^{-1}\hat{\mathbf{z}} - \frac{k}{N}\mathbf{e} \middle| -\hat{T}^{-1}\right). \end{aligned} \quad (182)$$

Thus

$$\begin{aligned} \frac{\Theta_k}{\Theta_{k-1}} &= \frac{\Theta(\hat{T}^{-1}\hat{\mathbf{z}} - \frac{k}{N}\mathbf{e} \middle| -\hat{T}^{-1})}{\Theta(\hat{T}^{-1}\hat{\mathbf{z}} - \frac{k-1}{N}\mathbf{e} \middle| -\hat{T}^{-1})} = e^{\frac{i\pi}{N}} \frac{\Theta\left[\begin{smallmatrix} \mathbf{e} \\ 0 \end{smallmatrix}\right]\left(\hat{T}^{-1}(\mathbf{z} + \frac{1}{2}\mathbf{e}) - \frac{k}{N}\mathbf{e} \middle| -\hat{T}^{-1}\right)}{\Theta\left[\begin{smallmatrix} \mathbf{e} \\ 0 \end{smallmatrix}\right]\left(\hat{T}^{-1}(\mathbf{z} + \frac{1}{2}\mathbf{e}) - \frac{k-1}{N}\mathbf{e} \middle| -\hat{T}^{-1}\right)} \equiv \\ &\equiv e^{\frac{i\pi}{N}} \frac{\Theta\left[\begin{smallmatrix} \mathbf{e} \\ 0 \end{smallmatrix}\right]\left(\hat{\mathbf{z}}^D - \frac{k}{N}\mathbf{e} \middle| -\hat{T}^{-1}\right)}{\Theta\left[\begin{smallmatrix} \mathbf{e} \\ 0 \end{smallmatrix}\right]\left(\hat{\mathbf{z}}^D - \frac{k-1}{N}\mathbf{e} \middle| -\hat{T}^{-1}\right)} \end{aligned} \quad (183)$$

where now

$$\hat{\mathbf{z}}^D = \hat{T}^{-1}(\mathbf{z} + \frac{1}{2}\mathbf{e})$$

and $\mathbf{e} \equiv (1, \dots, 1)$. The effect of the shift is to keep only the leading terms in the quadratic $(\mathbf{m} + \frac{1}{2}\mathbf{e})\hat{T}^{-1}(\mathbf{m} + \frac{1}{2}\mathbf{e})$ in the limit $\hat{T} \rightarrow 0$ yielding

$$\frac{\Theta_k}{\Theta_{k-1}} \stackrel{\hat{T} \rightarrow 0}{=} e^{\frac{i\pi}{N}} \frac{\vartheta_k(\hat{\mathbf{z}}^D)}{\vartheta_{k-1}(\hat{\mathbf{z}}^D)} \quad (184)$$

where

$$\vartheta_k(\hat{\mathbf{z}}^D) = \sum_{\mathbf{s} \in \mathbb{Z}_2^{N-1}} e^{i\pi \mathbf{s}(\mathbf{z}^D - \frac{k}{N} \mathbf{e}) - \frac{i\pi}{4} \sum_{j \neq j'} s_j \hat{T}_{jj'}^{-1} s_{j'}} \quad (185)$$

and $\mathbb{Z}_2 \equiv \{+1, -1\}$.

Using (173), (184) and identifying the off diagonal parts of \hat{T}^{-1} with T^D one can propose formulae for the Baker-Akhiezer functions

$$\Psi_k(\xi, t) \sim \xi^k e^{\sum_n t_n (\xi^n - \xi^{-n})} \frac{\vartheta_k(\hat{\mathbf{z}}^D(t_n; \xi))}{\vartheta_k(\hat{\mathbf{z}}^D(t_n))} \quad (186)$$

where

$$\vartheta_k(\hat{\mathbf{z}}^D(t_n)) = \sum_{m_j = \pm} e^{-i\pi \frac{k}{N} \sum_j m_j + i \sum_{j,n} m_j t_n \sin \frac{\pi j n}{N} + \frac{1}{4} \sum_{j \neq j'} m_j m_{j'} \log \frac{\sin \frac{\pi}{2N} (j+j')}{\sin \frac{\pi}{2N} |j-j'|}} \quad (187)$$

and

$$\begin{aligned} \vartheta_k(\hat{\mathbf{z}}^D(t_n; \xi)) &= \sum_{m_j = \pm} e^{-i\pi \frac{k}{N} \sum_j m_j + i \sum_{j,n} m_j t_n \sin \frac{\pi j n}{N} + \sum_{j \neq j'} m_j m_{j'} \log \frac{\sin \frac{\pi}{2N} (j+j')}{\sin \frac{\pi}{2N} |j-j'|}} \prod_j \left(\frac{\xi - \xi_j^-}{\xi - \xi_j^+} \right)^{m_j/2} \\ &= \sum_{m_j = \pm} e^{-i\pi \frac{k}{N} \sum_j m_j + i \sum_{j,n} m_j t_n \sin \frac{\pi j n}{N}} \prod_j \left(\frac{\xi - e^{-\frac{i\pi j}{N}}}{\xi - e^{\frac{i\pi j}{N}}} \right)^{m_j/2} \prod_{j \neq j'} \left(\frac{\sin \frac{\pi}{2N} (j+j')}{\sin \frac{\pi}{2N} |j-j'|} \right)^{\frac{m_j m_{j'}}{4}}. \end{aligned} \quad (188)$$

Low order cases of this are

- $SL(2)$. $N = 2$, $N - 1 = 1$, $m_j = m_1 = m = \pm$. Then

$$\vartheta_k(\hat{\mathbf{z}}^D(t_n; \xi)) = \sum_{m=\pm} e^{-\frac{i\pi k m}{2} + i m t} \left(\frac{\xi + i}{\xi - i} \right)^{m/2} = \frac{1}{\sqrt{\xi^2 + 1}} \left(e^{i(t - \frac{\pi k}{2})} (\xi + i) + e^{-i(t - \frac{\pi k}{2})} (\xi - i) \right) \quad (189)$$

- $SL(3)$. $N = 3$, $N - 1 = 2$, $j = 1, 2$, $(m_1, m_2) = \{(++), (+-), (-+), (--) \}$.

$$\begin{aligned} \vartheta_k(\hat{\mathbf{z}}^D(t_n; \xi)) &= \sum_{m_1, m_2 = \pm} e^{-\frac{i\pi k}{3} (m_1 + m_2) + i(m_1 z_1 + m_2 z_2)} \left(\frac{\xi - e^{-\frac{i\pi}{3}}}{\xi - e^{\frac{i\pi}{3}}} \right)^{m_1/2} \left(\frac{\xi - e^{-\frac{2\pi i}{3}}}{\xi - e^{\frac{2\pi i}{3}}} \right)^{m_2/2} 2^{\frac{m_1 m_2}{2}} \\ z_1 &= \sum t_n \sin \frac{\pi n}{3}, \quad z_2 = \sum t_n \sin \frac{2\pi n}{3}. \end{aligned} \quad (190)$$

These formulae give rise to the general form for the solitonic Baker-Akhiezer function of the periodic Toda chain

$$\Psi_k(\xi, t) = \xi^k e^{\sum_j t_j (\xi^j - \xi^{-j})} \frac{R_k(\xi, t)}{R(\xi)}. \quad (191)$$

Here $R(\xi)$ is a normalisation factor, independent of times, and chosen to be a polynomial of ξ of degree $N - 1$ in order for (191) to have desired analytic properties, while

$$R_k(\xi, t) = \psi_k(t) \prod_{s=1}^{N-1} (\xi - \mu_s(k, t)) = \sum_{l=0}^{N-1} r_l(k, t) \xi^l. \quad (192)$$

These Baker-Akhiezer functions are defined for $k : 0 \dots N - 1$ and extended to all k by

$$\Psi_{k+N} = w \Psi_k, \quad w = \xi^N \quad (193)$$

(equivalently $R_{k+N} = R_k$). Now the Toda chain Lax equation

$$\lambda \Psi_n = C_{n+1} \Psi_{n+1} + p_n \Psi_n + C_n \Psi_{n-1}, \quad C_n \equiv e^{\frac{1}{2}(q_n - q_{n-1})}, \quad \lambda = \xi + \frac{1}{\xi} \quad (194)$$

implies that

$$\begin{aligned} r_0(n) - C_n r_0(n-1) &= 0 \\ r_1(n) - C_n r_1(n-1) - p_n r_0(n) &= 0 \end{aligned} \quad (195)$$

and so

$$r_0(n) = C_n r_0(n-1) = \dots = e^{\frac{1}{2}(q_n - q_0)} r_0(0) \sim e^{\frac{1}{2}q_n}. \quad (196)$$

For the solitons coming from degeneration of N -periodic Toda chain one should also impose the “gluing conditions”

$$\Psi_n(\xi_j) = \Psi_n\left(\frac{1}{\xi_j}\right) \quad j = 1, \dots, N-1. \quad (197)$$

This means that the Baker-Akhiezer function remembers that it originally came from a genus $N-1$ Riemann surface, and each pair of points $\xi_j, \frac{1}{\xi_j}$ corresponds to a degenerate handle. Now the conditions (197) together with (193) entail

$$\xi_j^{2N} = 1. \quad (198)$$

Thus we may take

$$\xi_j = e^{\frac{i\pi j}{N}} \quad (199)$$

where the label j can be restricted to $j = 1, \dots, N-1$ since

$$\phi_j = \xi_j + \frac{1}{\xi_j} = 2 \cos \frac{\pi j}{N} = \phi_{2N-j}. \quad (200)$$

Eq. (197) explicitly reads

$$\frac{R_n(\frac{1}{\xi_j})}{R_n(\xi_j)} = \prod_{k=1}^{N-1} \frac{\xi_j^{-1} - \mu_k(n, t)}{\xi_j - \mu_k(n, t)} = e^{\frac{2\pi i n j}{N} + 4i \sum_l t_l \sin \frac{\pi j l}{N} + Z_j(R)}, \quad j = 1, \dots, N-1. \quad (201)$$

Here $Z_j(R) = \log \frac{R(\frac{1}{\xi_j})}{R(\xi_j)}$, thus if one has chosen $R(\xi) = \prod_{s=1}^{N-1} (\xi - \gamma_s)$ in (191) then $Z_j(R) = \sum_{s=1}^{N-1} \log \frac{\frac{1}{\xi_j} - \gamma_s}{\xi_j - \gamma_s}$. Now (201) is a system of linear equations for the coefficients $r_k(n, t)$ of the polynomial $R_n(\xi, t)$:

$$\sum_{k=0}^{N-1} \sin \left(\frac{\pi j k}{N} + \frac{\pi j n}{N} + 2 \sum_l t_l \sin \frac{\pi j l}{N} - \frac{i}{2} Z_j(R) \right) r_k(n, t) = 0. \quad (202)$$

Such can be readily solved. Conditions (200) can be interpreted as values of the scalar fields at the critical points of the superpotential, while the soliton trajectories connect the critical points.

4.2 On the “solitonic” limit of the elliptic Calogero-Moser system

The solitonic limit of the elliptic Calogero-Moser system involves many open questions, though it seems that we can proceed in an analogous manner to the Toda chain case. We will discuss here only some explicit computations for low N and will postpone the computation of the soliton phases or string tensions.

For the $N = 2$ case the solitonic limit corresponds to $h = e = \wp(\omega)$ (ω may be any (!) half-period) so that (65) turns into

$$\lambda^2 - x + h = \lambda^2 - \wp(z) + \wp(\omega) = (\lambda - \Phi(z, \omega))(\lambda + \Phi(z, \omega)) = 0. \quad (203)$$

Thus

$$\lambda = \pm \Phi(z, \omega) = \pm \frac{\sigma(z - \omega)}{\sigma(z)\sigma(\omega)} e^{\zeta(\omega)z} = \mp \frac{\sigma(z + \omega)}{\sigma(z)\sigma(\omega)} e^{-\zeta(\omega)z} \quad (204)$$

and these functions are related to the entries of the Calogero-Moser Lax operator (40) at half-periods via

$$\begin{aligned} F(\omega|z) &= \frac{\sigma(z + \omega)}{\sigma(z)\sigma(\omega)} = \Phi(z, \omega) e^{\zeta(\omega)z} \\ F(-\omega|z) &= -\frac{\sigma(z - \omega)}{\sigma(z)\sigma(\omega)} = \frac{\sigma(z + \omega)}{\sigma(z)\sigma(\omega)} e^{-2\zeta(\omega)z} = \Phi(z, \omega) e^{-\zeta(\omega)z}. \end{aligned} \quad (205)$$

In this limit one also has

$$y^2 = (x - e)(x - e_+)(x - e_-) = \lambda^2(\lambda^2 - \tilde{e}_+)(\lambda^2 - \tilde{e}_-), \quad \tilde{e}_{\pm} = e_{\pm} - e. \quad (206)$$

The Seiberg-Witten differential and periods are

$$dS = \lambda \frac{dx}{y} = \frac{dx}{\sqrt{(x - e_+)(x - e_-)}}, \quad \oint_B dS = 0, \quad \oint_A dS = 2\pi i. \quad (207)$$

Thus instead of $\Sigma_{\text{reduced}}^{CM}$ (113) one may introduce the *rational* reduced curve

$$Y^2 = (x - e_+)(x - e_-). \quad (208)$$

Indeed, a direct integration of the equation of motion (114) gives now

$$2it = \int \frac{dx}{(x - e)\sqrt{(x - e_+)(x - e_-)}}, \quad (209)$$

a contour integral on the *rational* curve (208).

Let us note that formula (203) can be obtained by considering the extrema of the Calogero-Moser hamiltonian (the superpotential in the SW approach). Then

$$dh = \frac{\partial h}{\partial p} dp + \frac{\partial h}{\partial q} dq = 0 \quad (210)$$

yields

$$p = 0, \quad \wp'(q) = 0. \quad (211)$$

The latter is satisfied by *any* half-period $q = \omega$, so that

$$h|_{dh=0} = \wp(\omega) = e. \quad (212)$$

A similar argument holds for the general N -particle case. Now the Lax operator (39) computed at the “stationary” points yields $p_1 = \dots = p_N = 0$ and the locus equation

$$\mathcal{L}_N = \{q_j | q_j \neq q_i, \sum_{j \neq i} \wp'(q_{ij}) = 0, i : 1 \dots N\}. \quad (213)$$

The (closure of) this locus has a rich geometry and many questions regarding it are still unanswered. The early work of [34] is still one of the most detailed investigations (see also [25]). The (closure of) the locus has in general several disconnected pieces some of which are trivial copies of the torus. The latter are easily understood: any odd, periodic function (with period L) satisfies the identity

$$f\left(\frac{L}{N}\right) + f\left(\frac{2L}{N}\right) + f\left(\frac{3L}{N}\right) + \dots + f\left(\frac{N-2}{N}L\right) + f\left(\frac{N-1}{N}L\right) = 0 \quad (214)$$

for any odd N and for N even provided $f\left(\frac{L}{2}\right) = 0$. Choosing $\wp'(x)$ as the function $f(x)$ with $L = 2\omega$ (and $\wp'(\omega) = 0$) one gets a solution to (213) with

$$q_k = q_0 + \frac{2\omega}{N}k, \quad q_{jk} = \frac{2\omega}{N}(j - k). \quad (215)$$

The loci here are simple copies of the torus; by varying the periods in this construction one gets further simple solutions. There are however other solutions less well understood.

Lets consider the $N = 3$ case. For $N = 3$ eq. (213) is equivalent to

$$\wp'(q_{12}) = \wp'(q_{23}) = -\wp'(q_{13}). \quad (216)$$

This has solutions (together with $q_{12} + q_{23} = q_{13}$)

$$q_{12} = \omega, \quad q_{23} = \omega', \quad q_{13} = \omega + \omega' \quad (217)$$

and

$$q_{12} = \frac{2\omega}{3}, \quad q_{23} = \frac{2\omega}{3}, \quad q_{13} = \frac{4\omega}{3} = -\frac{2\omega}{3} \quad (218)$$

for *any* half-period $\omega = (\omega, \omega', \omega + \omega')$ ¹². Upon substituting (218) and $p_i = 0$ into the Lax equation (39) one gets

$$\lambda^3 + 3F_+F_- \lambda - F_+^3 + F_-^3 = 0 \quad (220)$$

with the three solutions

$$\lambda_0 = F_+ - F_-, \quad \lambda_{\pm} = e^{\pm \frac{2\pi i}{3}} F_+ + e^{\pm \frac{i\pi}{3}} F_-. \quad (221)$$

Here

$$F_{\pm} = \frac{\sigma\left(z \pm \frac{2\omega}{3}\right)}{\sigma(z)\sigma\left(\frac{2\omega}{3}\right)} e^{\mp \frac{2}{3}\zeta(\omega)z}. \quad (222)$$

¹²One also has the relation between ζ -functions $\eta = \zeta(\omega)$ in the half-periods

$$\zeta(\omega + \omega') = \zeta(\omega) + \zeta(\omega') \quad (219)$$

5 Discussion

In this paper we have considered the singular limits of various SW integrable systems that are relevant for the weak and strong coupling limits of the corresponding field theories. We conclude by making several comments about the relation of these relatively straightforward calculations in integrable models with corresponding properties of the SUSY gauge theories.

Let us begin with the outstanding problem of why there is a correspondence at all between integrable systems and SW theory. Part of the problem is that, from a purely four-dimensional perspective, Seiberg-Witten theory only sees the commuting Hamiltonians of a mechanical system. Only these quantities appear as coefficients of the Seiberg-Witten curve. Half of phase space is not apparent at all and the choice of mechanical system appears arbitrary. We believe viewing these systems from a three dimensional perspective sheds light on the matter. In fact one should also pay attention to the four dimensional $\mathcal{N} = 2$ gauge theories compactified on $\mathbb{R}^3 \times S^1$ [35]. Recall that the phase spaces of integrable systems play the role of moduli spaces, or spaces of vacua parameters, of (compactified) SUSY gauge theories in the following way. The minima of the scalar potential in the gauge theories with extended supersymmetry

$$V(\Phi) = \text{Tr} \sum_{I < J} [\Phi^I, \Phi^J]^2 \quad (223)$$

correspond to simultaneously diagonalisable matrices $[\Phi^I, \Phi^J] = 0$ whose eigenvalues $\{\phi_k^I\}$ can be thought of as the (complexified) momenta of some “particles”. Now for theories with a compact dimension one should also add “co-ordinates” $\{q_k^{\mathbb{R}}\}$ corresponding to the eigenvalues of the Wilson loops $\oint A_\mu dx^\mu$. For $\mathcal{N} = 2$ vector multiplets in four dimensions¹³ one has a single complex scalar and compactification of one space-time dimension gives rise to an extra complex scalar $q = q^{\mathbb{R}} + i\gamma$, where $q^{\mathbb{R}}$ comes from the set of eigenvalues of the Wilson loop $\oint A_3$ and γ corresponds to the (set of) 3D dual photons $\partial_\alpha A_\beta = \epsilon_{\alpha\beta\rho} \partial_\rho \gamma$, $\alpha, \beta = 0, \dots, 2$. Thus by viewing the theory on $\mathbb{R}^3 \times S^1$ one may naturally include coordinates. The moduli space of vacua for the theories on $\mathbb{R}^3 \times S^1$ is a hyper-Kähler manifold, and, we recall, such are the phase spaces of algebraically completely integrable systems. Further, it was argued [35] that there is a distinguished complex structure on this moduli space independent of the radius of the compact S^1 . This yields the complex structure of the mechanical system. In general we expect the symplectic structure to arise from a careful treatment of the central charges of SUSY algebra:

$$\Omega \sim Z^{BPS} \sim \int_{d\sigma_{jk}} \text{Tr}(F_{jk} + i\tilde{F}_{jk})\Phi \sim \text{Tr} \int \delta A \wedge \delta \Phi \sim \sum \delta q_i \wedge \delta p_i. \quad (224)$$

For a $\mathcal{N} = 4$ theory, we have *three* different choices for the symplectic structure (224), corresponding to the three different scalar fields in the $\mathcal{N} = 4$ theory, and these are related by a “hyper-Kähler” rotation.

Now consider the decompactification of the “3D” gauge theory on $\mathbb{R}^3 \times S^1$ with symplectic form (224) in the limit of the S^1 radius $R \rightarrow \infty$. The dimension of the moduli space of the $\mathbb{R}^3 \times S^1$ theory is twice that of the limiting 4D theory: the “coordinates” are no longer present. The resulting 4D gauge theory is associated to an integrable system: the 4D moduli are, from the 3D point of view, the Poisson-commuting *hamiltonians* constructed from the 3D momenta and co-ordinates, with respect to symplectic structure (224). From this perspective, quantum effects in the compactified $\mathcal{N} = 2$ gauge theory turn the bare symplectic form (224) into

$$\sum \delta q_i \wedge \delta p_i \mapsto \sum \delta z_i \wedge \delta a_i, \quad (225)$$

where the SW integrals [1]

$$a_i = \oint_{A_i} dS \quad (226)$$

are the correct quantum variables. One may consider $a_i = a_i(\Phi, \Lambda)$ as a transformation from the bare quantities $\{\phi_i\}$ to their exact quantum values $\{a_i\}$ playing the role of the quantum BPS masses of the effective theory. In the same way one should consider the transformation $q_i \rightarrow z_i$ as transformation from bare values of the monodromy to the exact quantum values of the effective theory.

For theories with four-dimensional $\mathcal{N} = 4$ SUSY the effective couplings and BPS masses (i.e. the eigenvalues of the scalar fields $\{\phi_i\}$ and Wilson loops) are *not renormalised*, since the symmetries of the theory include

¹³An $\mathcal{N} = 2$ 4D vector supermultiplet in the adjoint representation consists of an $\mathcal{N} = 1$ 4D vector multiplet (A_μ, ψ) together with an $\mathcal{N} = 1$ 4D scalar multiplet (ϕ, χ) . Here ψ and χ are two complex Weyl spinors. If say χ acquires a nontrivial phase (231) under a shift along the loop in the compact direction, it becomes massive with the mass $\frac{\epsilon}{R}$. The 4D $\mathcal{N} = 1$ vector multiplet remains massless, and can be represented by a 3D $\mathcal{N} = 2$ supermultiplet $(A_\alpha, \psi, \frac{q}{R})$, where $\alpha = 0, 1, 2$, $q = RA_3$ and ψ is 3D complex spinor.

the conformal group and so there are no dimensionful scale or mass parameters. The corresponding integrable model is a system of *free* particles: the Hamiltonians are $u_k = \frac{1}{k} \text{Tr} \Phi^k = \frac{1}{k} \sum_i p_i^k$, where $\{p_i\}$ play the role of momenta and the co-ordinates coming from the “compact” moduli depend linearly on the angles. Breaking four dimensional supersymmetry down to $\mathcal{N} = 2$ (for example, by adding extra mass terms $m_i^2 \text{Tr} \Phi_i^2$ for two of the three scalar fields) reduces the dimension of the “scalar” moduli space down to $2N$, that of one complex “diagonal matrix” field. Moreover, in contrast to the $\mathcal{N} = 4$ theory, the matrices of the scalar fields and the monodromies become dependent upon each other, or satisfy a nontrivial commutation relation (coming from the vanishing of D-terms)

$$[A, \Phi] \sim mJ \quad (227)$$

in the general case (of nontrivial boundary conditions). Here J is some matrix of “gauge-covariant” form and the right hand side here is linear in the parameter of the “massive deformation” [5, 6], (for $m \rightarrow 0$ one comes back to $\mathcal{N} = 4$ theory).

Let us consider the compactification of an $\mathcal{N} = 2$ SUSY Yang-Mills theory with just a vector supermultiplet to $\mathbb{R}^3 \times S^1$ (with S^1 having radius R) in more detail.¹⁴ If one takes all the fields to have *periodic* boundary conditions in the compact direction this would yield an $\mathcal{N} = 4$ (in the three-dimensional sense) SUSY theory. If, however, one puts [36, 13]

$$\phi(x + R) = e^{i\epsilon} \phi(x) \quad (231)$$

on *half* of the fields, the resulting theory would have only $\mathcal{N} = 2$ *three*-dimensional SUSY ($\mathcal{N} = 1$ in the four-dimensional sense), i.e. the supersymmetry will be (partially) broken by the non-periodic boundary conditions. Now in contrast to $\mathcal{N} = 4$ SUSY in 3D, an $\mathcal{N} = 2$ supersymmetric theory can generate a superpotential [37]. In terms of the complexified variables $q = q^{\mathbb{R}} + i\gamma$, the superpotential acquires the form [35, 38]

$$W \sim \epsilon \text{Tr} \Phi^2 + \frac{\epsilon}{R^2} \left(\sum_{i=1}^{N-1} e^{q_{i+1} - q_i} + e^{q_1 - q_N} \right). \quad (232)$$

Here the first term is the “4D contribution” while the second term (the first term in the brackets) has a 3D origin; the final term is induced by 3+1D instanton contributions. The simple roots of the second term are the usual 3D instantons (4D BPS “monopoles”) and give the potential of the *open* Toda chain: $\Pi_2\left(\frac{SU(N)}{U(1)^{N-1}}\right) \cong \mathbb{Z}^{N-1}$. The final term (minus the highest root) appears only in 3+1 dimensions and can be treated as a 4D instanton (or caloron) contribution. In the perturbative limit, considered in detail in this paper, one may directly see the nonrenormalisability of the superpotential, which means, in particular, that

$$W^{4D} = h = a^2 = p^2 + e^{2q} = W^{3D}. \quad (233)$$

Now let us discuss how the formulas obtained in this paper may be reinterpreted from this three-dimensional perspective. First, we have discussed the singular limits of the SW integrable systems, when the spectral curves degenerate and become rational with marked points (i.e. their “smooth” genus is zero – eqs. (1), (4), (8), (9), (158), (163) and (208)). Degeneration means that the discriminant of the corresponding smooth curve vanishes as one approaches particular (boundary) points of the moduli space. In the paper above we have divided these singular limits into two groups: weak-coupling from the point of view of the SUSY gauge

¹⁴In practice this means for the theory at mass scale $\Lambda = \Lambda_{QCD}$ that $\Lambda \gg \frac{1}{R}$ corresponds to a 4D theory while $\Lambda \ll \frac{1}{R}$ to a 3D theory. For the couplings one has $\frac{1}{g_3^2} = \frac{R}{g_4^2}$, i.e. the 3D theory with fixed coupling g_3 corresponds to $R \rightarrow 0$, or $g_4 \rightarrow 0$ and so to the weak coupling limit of the 4D theory. In this limit the 4D instantons decouple since the 4D coupling g_4 is small (for $R \rightarrow 0$ and $g_3 = \text{fixed}$) and so that the exponential terms, powers of $e^{-\frac{1}{g_4^2}}$, may be neglected. In the 3D theory the integral for the polarization operator

$$(k_\mu k_\nu - \delta_{\mu\nu} k^2) \left| \frac{1}{m} \right| + \dots \quad (228)$$

yields a convergent integral by dimensional arguments. In the “intermediate” case of a compactified (3+1)-dimensional theory on $\mathbb{R}^3 \times S^1$ with finite S^1 radius R one can present this result in terms of a 3D theory together with a sum of Kaluza-Klein contributions

$$\frac{1}{g_3^2} = \sum_n \left| \frac{1}{m + \frac{n}{R}} \right| \quad (229)$$

For $R \rightarrow 0$ only the term with $n = 0$ survives in the sum (229) leading back to the 3D expression. For the opposite limit $R \rightarrow \infty$ one can define the dimensionless 4D coupling and replace the (divergent) sum by an integral

$$\frac{1}{g_4^2} = \frac{1}{R g_3^2} = \frac{1}{R} \sum_{|n| \leq |n_{\max}|} \left| \frac{1}{m + \frac{n}{R}} \right| = \int_m^{\frac{n_{\max}}{R} \equiv \Lambda} \frac{d\mu}{\mu} = \log \frac{\Lambda}{m}. \quad (230)$$

theory which yields “open” (or “trigonometric”) integrable systems; and strong coupling yielding “solitonic” behaviour. If one deals with the gauge theory with *extended* supersymmetry each of these degenerate curves as well as the smooth ones play equivalent roles as physical vacua. By breaking the extended supersymmetry down to $\mathcal{N} = 1$ (in the four-dimensional sense) however, one generates integrable dynamics *distinguishing* the second, “solitonic”, group of degenerate curves from those of the “perturbative” ones, so that only the solitonic points play the role of vacua in $\mathcal{N} = 1$ theories. It is clear from the explicit form of solitonic solutions obtained here that the angles, corresponding to Wilson loops, are proportional to the parameters of the partial SUSY violation (cf. eqs. (171) and (190) with refs. [31, 32]). For both kinds of singular limits we have shown how explicit solutions may be straightforwardly calculated. The co-ordinates and momenta of the integrable systems of particles (the classical 3D moduli) are expressed through the theta-functions of the SW curves. In the singular limits these theta-functions degenerate into the *finite* sums – see eqs. (123), (124), (133), (136), (161), (187) – suggestive of a kind of partition function for a “discretized” matrix models. The “dual” point of view is to consider the theta-functions Θ_k as generating functions for Hamiltonians of a dual integrable systems with co-ordinate’s a_i and momenta z_i . The Poisson commutativity of their ratios, implicitly proven in [26], can be checked by straightforward calculation in the degenerate limits. The fact that degenerate theta-functions appear in the form of “discretized” matrix models (133), where instead of an integral over the real line one has a (multiple) discrete sum over the *spectrum* of the Toda molecule, may be considered as a possible (though very speculative) sign of a dual description of SW theory in the language of quantum gravity, possibly in a “holographic” sense.

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A Elliptic Functions and the Inozemtsev Limit

In this appendix we shall review some basic definitions and properties of elliptic functions and then consider the Inozemtsev limit.

The Weierstrass elliptic function $\wp(z|\omega, \omega')$ is the doubly periodic function with periods $2\omega, 2\omega'$ given by

$$\wp(z|\omega, \omega') = \frac{1}{z^2} + \sum_{n,m \in \mathbb{Z}}' \left(\frac{1}{(z + 2\omega n + 2\omega' m)^2} - \frac{1}{(2\omega n + 2\omega' m)^2} \right). \quad (234)$$

It satisfies the differential equation

$$\wp(z)'^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3), \quad (235)$$

and scaling relation

$$\wp(tz|t\omega, t\omega') = t^{-2} \wp(z|\omega, \omega'). \quad (236)$$

Using

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} \right), \quad \sum_{n \in \mathbb{Z}} \frac{1}{(x+n)^2} = \frac{\pi^2}{\sin^2 \pi x}$$

we may write

$$\wp(\tilde{z}|\frac{1}{2}, \frac{\tau}{2}) = \sum_{m \in \mathbb{Z}} \frac{\pi^2}{\sin^2 \pi(\tilde{z} + m\tau)} - \frac{\pi^2}{3} - \sum_{m \in \mathbb{Z}}' \frac{\pi^2}{\sin^2 \pi m\tau} \quad (237)$$

where $\tilde{z} = \frac{z}{2\omega}$. A slight rewriting of this utilising the scaling formula yields

$$\wp(v|i\pi\tau, i\pi) = \frac{1}{12} - \frac{1}{2} \sum_{m>0} \frac{1}{\sinh^2(i\pi m\tau)} + \frac{1}{2} \sum_{k=-\infty}^{\infty} \frac{1}{\cosh(v + 2i\pi k\tau) - 1}. \quad (238)$$

This is uniformly convergent for $Im(\tau) > 0$.

The roots of e_i of equation (235) may be expressed in terms of theta functions by

$$\begin{aligned}
e_1 &= \frac{\pi^2}{12\omega^2} (\theta_3(0)^4 + \theta_4(0)^4) \\
&= \frac{\pi^2}{12\omega^2} \prod_{n=1}^{\infty} (1 - q^{2n})^4 (\prod_{n=1}^{\infty} (1 + q^{2n-1})^8 + \prod_{n=1}^{\infty} (1 - q^{2n-1})^8) \\
e_2 &= \frac{\pi^2}{12\omega^2} (\theta_2(0)^4 - \theta_4(0)^4) \\
&= \frac{\pi^2}{12\omega^2} \prod_{n=1}^{\infty} (1 - q^{2n})^4 (16q \prod_{n=1}^{\infty} (1 + q^{2n})^8 - \prod_{n=1}^{\infty} (1 - q^{2n-1})^8) \\
e_3 &= -\frac{\pi^2}{12\omega^2} (\theta_2(0)^4 + \theta_3(0)^4) \\
&= -\frac{\pi^2}{12\omega^2} \prod_{n=1}^{\infty} (1 - q^{2n})^4 (16q \prod_{n=1}^{\infty} (1 + q^{2n})^8 + \prod_{n=1}^{\infty} (1 - q^{2n-1})^8)
\end{aligned} \tag{239}$$

where $q = e^{i\pi\tau}$ and $\tau = \frac{\omega'}{\omega}$. As $q \rightarrow 0$ we have

$$e_1 = \frac{\pi^2}{6\omega^2} + O(q^2), \quad e_2 = -\frac{\pi^2}{12\omega^2} + \frac{\pi^2}{12\omega^2} 16q + O(q^2), \quad e_3 = -\frac{\pi^2}{12\omega^2} - \frac{\pi^2}{12\omega^2} 16q + O(q^2).$$

Thus in this limit

$$\begin{aligned}
e_1 &\equiv e = \frac{\pi^2}{6\omega^2} \\
e_2 &= e_3 = e_{\pm} = -\frac{\pi^2}{12\omega^2} \\
x &= \wp(z) = -\frac{\pi^2}{12\omega^2} + \frac{\pi^2}{4\omega^2} \frac{1}{\sin^2 \pi \tilde{z}} \equiv \frac{\pi^2}{4\omega^2} (\tilde{x} - \frac{1}{3}) \\
y &= -\frac{1}{2} \wp'(z) = \frac{\pi^3}{8\omega^3} \frac{\cos \pi \tilde{z}}{\sin^3 \pi \tilde{z}} \equiv \frac{\pi^3}{8\omega^3} \tilde{y} \\
\tilde{y}^2 &= \tilde{x}^2 (\tilde{x} - 1)
\end{aligned} \tag{240}$$

Consider now $m^2(\wp(v) - \wp(z))$. The Inozemtsev limit is a double scaling limit in that the coupling constant m here is scaled as well as the period of the \wp function. Let

$$m = M e^{-i\pi\tau/2}, \quad v = \phi - i\pi\tau, \quad w = e^{z+i\pi\tau}, \tag{241}$$

then

$$\begin{aligned}
m^2(\wp(v) - \wp(z)) &= \frac{M^2}{2} \sum_{k=-\infty}^{\infty} \left(\frac{e^{-i\pi\tau}}{\cosh(\phi + i\pi(2k-1)\tau) - 1} - \frac{e^{-i\pi\tau}}{\cosh(z + 2i\pi k\tau) - 1} \right) \\
&\xrightarrow{Im(\tau) \rightarrow \infty} 2M^2 \cosh \phi - M^2 \left(w + \frac{1}{w} \right).
\end{aligned} \tag{242}$$

B Proof of the Edelstein-Mas Conjecture

We have

$$-2i\pi \tilde{T}_{ij}^{\text{pert}} = \delta_{ij} \sum_{l \neq i} \log(a_i - a_l)^2 - (1 - \delta_{ij}) \log(a_i - a_j)^2. \tag{243}$$

Let us denote by

$$S = \sum_{k=0}^{N-1} \sin \frac{\pi k i'}{N} \sin \frac{\pi k j'}{N} \sum_{i,j=1}^N \tilde{T}_{ij}^{\text{pert}} (a_l \rightarrow 2 \cos \frac{\pi(l - \frac{1}{2})}{N}) \cos \frac{\pi k(i - \frac{1}{2})}{N} \cos \frac{\pi k(j - \frac{1}{2})}{N} \tag{244}$$

The Edelstein-Mas conjecture is (for $i' \neq j'$)

$$S \stackrel{???}{=} \frac{N^2}{4} T_{i'j'}^D \tag{245}$$

and we shall now establish this. We have shown that a direct calculation yields (for $j \neq k$)

$$T_{jk}^D = \oint_{A_j} d\omega_k^D = \frac{1}{2\pi i} \log \frac{\sin^2 \frac{\pi}{2N} (j - k)}{\sin^2 \frac{\pi}{2N} (j + k)} = \frac{1}{i\pi} \log \frac{\sin \frac{\pi}{2N} |j - k|}{\sin \frac{\pi}{2N} (j + k)}. \tag{246}$$

To proceed we first perform the sum over k ,

$$\sum_{k=0}^{N-1} \sin \frac{\pi k i'}{N} \sin \frac{\pi k j'}{N} \cos \frac{\pi k(i-\frac{1}{2})}{N} \cos \frac{\pi k(j-\frac{1}{2})}{N} = \frac{1}{8} \sum_{\alpha \in \Delta_+} \sum_{k=0}^{N-1} \cos\left(\frac{\pi k \alpha}{N}\right) - \frac{1}{8} \sum_{\alpha \in \Delta_-} \sum_{k=0}^{N-1} \cos\left(\frac{\pi k \alpha}{N}\right)$$

where

$$\Delta_+ = \{i' - j' \pm (i - j), i' - j' \pm (i + j - 1)\}, \quad \Delta_- = \{i' + j' \pm (i - j), i' - j' \pm (i + j - 1)\}.$$

Now

$$\sum_{k=0}^{N-1} \cos\left(\frac{\pi k \alpha}{N}\right) = \begin{cases} \frac{1}{2}[1 - \cos \pi \alpha] & \alpha \neq 0, 2N, \\ N & \alpha = 0, 2N. \end{cases}$$

If every $\alpha \in \Delta_{\pm}$ is distinct from 0 or $2N$, there is cancelling between the terms in the sums. We find (taking account of parity) that

$$\sum_{k=0}^{N-1} \sin \frac{\pi k i'}{N} \sin \frac{\pi k j'}{N} \cos \frac{\pi k(i-\frac{1}{2})}{N} \cos \frac{\pi k(j-\frac{1}{2})}{N} = \frac{N}{8} \left(\sum_{\alpha \in \Delta_+} \delta_{\alpha,0} + \delta_{\alpha,2N} - \sum_{\alpha \in \Delta_-} \delta_{\alpha,0} + \delta_{\alpha,2N} \right).$$

Without loss of generality we may assume $i' > j'$. Then

$$\begin{aligned} \sum_{i,j=1}^N \sum_{\alpha \in \Delta_+} (\delta_{\alpha,0} + \delta_{\alpha,2N}) \tilde{T}_{ij}^{\text{pert}} &= \sum_{i=1}^{N-(i'-j')} \tilde{T}_{i i+(i'-j')}^{\text{pert}} + \sum_{i=i'-j'+1}^N \tilde{T}_{i i-(i'-j')}^{\text{pert}} \\ &+ \sum_{i=N-(i'-j')+1}^N \tilde{T}_{i 2N-i-(i'-j')+1}^{\text{pert}} + \sum_{i=1}^{i'-j'} \tilde{T}_{i i'-j'+1-i}^{\text{pert}} \end{aligned} \quad (247)$$

In the limit $a_l \rightarrow 2 \cos \frac{\pi(l-\frac{1}{2})}{N}$ we note that

$$\tilde{T}_{ij}^{\text{pert}} = \tilde{T}_{ij+2N}^{\text{pert}} = \tilde{T}_{i1-j}^{\text{pert}} = \tilde{T}_{i2N+1-j}, \quad (248)$$

and for $i \neq j$

$$\begin{aligned} \tilde{T}_{ij}^{\text{pert}} &= \frac{1}{2i\pi} \log 4 \left(\cos \frac{\pi(i-\frac{1}{2})}{N} - \cos \frac{\pi(j-\frac{1}{2})}{N} \right)^2 \\ &= \frac{1}{2i\pi} \log \left(16 \sin^2 \frac{\pi(i-j)}{2N} \sin^2 \frac{\pi(i+j-1)}{2N} \right) \end{aligned} \quad (249)$$

To proceed we now distinguish two cases depending upon whether $i' - j'$ is even or odd. Let us take the even case first. Using (248) we may combine the first and third terms of the right hand side, of (247) and the second and last terms to give

$$\sum_{i,j=1}^N \sum_{\alpha \in \Delta_+} (\delta_{\alpha,0} + \delta_{\alpha,2N}) \tilde{T}_{ij}^{\text{pert}} = \sum_{i=1}^N \tilde{T}_{i i+(i'-j')}^{\text{pert}} + \sum_{i=1}^N \tilde{T}_{i i-(i'-j')}^{\text{pert}}. \quad (250)$$

The sum over Δ_- similarly simplifies, though we note that here there are now distinct cases to be considered (depending on whether $i' + j' < N$ or not). We obtain

$$\sum_{i,j=1}^N \sum_{\alpha \in \Delta_-} (\delta_{\alpha,0} + \delta_{\alpha,2N}) \tilde{T}_{ij}^{\text{pert}} = \sum_{i=1}^N \tilde{T}_{i i+(i'+j')}^{\text{pert}} + \sum_{i=1}^N \tilde{T}_{i i-(i'+j')}^{\text{pert}} \quad (251)$$

Combining our results shows (for $i' \neq j'$)

$$\begin{aligned} S &= \frac{N}{8} \sum_{i=1}^N \left(\tilde{T}_{i i+(i'-j')}^{\text{pert}} + \tilde{T}_{i i-(i'-j')}^{\text{pert}} - \tilde{T}_{i i+(i'+j')}^{\text{pert}} - \tilde{T}_{i i-(i'+j')}^{\text{pert}} \right) = \\ &= \frac{N}{16i\pi} \log \prod_{i=1}^N \left(\frac{\sin^4 \frac{\pi(i'-j')}{2N} \sin^2 \frac{\pi(2i+i'-j'-1)}{2N} \sin^2 \frac{\pi(2i-i'+j'-1)}{2N}}{\sin^4 \frac{\pi(i'+j')}{2N} \sin^2 \frac{\pi(2i+i'+j'-1)}{2N} \sin^2 \frac{\pi(2i-i'+j'-1)}{2N}} \right) = \\ &= \frac{N^2}{4} T_{i'j'}^D \end{aligned} \quad (252)$$

thus proving the conjecture for this case. Observe that no terms in this product vanish with our assumption of $i' - j'$ being even.

The remaining case follows in an analogous fashion, the difference now being that T_{ii} terms can appear when $i' - j'$ is odd. Set $\delta = i' - j'$ and $\bar{\delta} = i' + j'$. Let $\bar{\delta} < N$ (a similar argument holding for $\bar{\delta} > N$). With δ odd (250) takes the form

$$\sum_{i,j=1}^N \sum_{\alpha \in \Delta_+} (\delta_{\alpha,0} + \delta_{\alpha,2N}) \tilde{T}_{ij}^{\text{pert}} = \sum_{i=1}^{N'} \tilde{T}_{ii+\delta}^{\text{pert}} + \sum_{i=1}^{N'} \tilde{T}_{ii-\delta}^{\text{pert}} + \tilde{T}_{\frac{\delta+1}{2}, \frac{\delta+1}{2}}^{\text{pert}} + \tilde{T}_{N-\frac{\delta-1}{2}, N-\frac{\delta-1}{2}}^{\text{pert}}, \quad (253)$$

while (251) becomes

$$\sum_{i,j=1}^N \sum_{\alpha \in \Delta_-} (\delta_{\alpha,0} + \delta_{\alpha,2N}) \tilde{T}_{ij}^{\text{pert}} = \sum_{i=1}^N \tilde{T}_{ii+\delta}^{\text{pert}} + \sum_{i=1}^N \tilde{T}_{ii-\delta}^{\text{pert}} + \tilde{T}_{\frac{\delta+1}{2}, \frac{\delta+1}{2}}^{\text{pert}} + \tilde{T}_{N-\frac{\delta-1}{2}, N-\frac{\delta-1}{2}}^{\text{pert}}. \quad (254)$$

Combining these two expressions yields

$$S = \frac{N(N-1)}{4} T_{i'j'}^D + \frac{N}{8} \left(\tilde{T}_{\frac{\delta+1}{2}, \frac{\delta+1}{2}}^{\text{pert}} + \tilde{T}_{N-\frac{\delta-1}{2}, N-\frac{\delta-1}{2}}^{\text{pert}} - \tilde{T}_{\frac{\bar{\delta}+1}{2}, \frac{\bar{\delta}+1}{2}}^{\text{pert}} - \tilde{T}_{N-\frac{\bar{\delta}-1}{2}, N-\frac{\bar{\delta}-1}{2}}^{\text{pert}} \right).$$

Finally the last four terms in brackets may be simplified (with most terms cancelling) leaving $2T_{i'j'}^D$. Again we arrive at

$$S = \frac{N^2}{4} T_{i'j'}^D,$$

proving the conjecture.

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